4 Linear Programming (LP)

Definition: A *Linear Programming* (LP) problem is an optimization problem:

$$\begin{array}{ll} \min & f(\underline{x}) \\ s.t. & \\ & \underline{x} \in X \subseteq \mathbb{R}^n \end{array}$$

where

- the *objective function* $f : X \to \mathbb{R}$ is *linear*,
- the *feasible region* $X = \{ \underline{x} \in \mathbb{R}^n : g_i(\underline{x}) \text{ r}_i \ 0, i \in \{1,\ldots, m\} \}$ with $\text{r}_i \in \{=, \geq, \leq\}$ and $g_i : \mathbb{R}^n \to \mathbb{R}$ *linear* functions $\forall \ i = 1,\ldots, m$.

Definition: $\underline{x}^* \in \mathbb{R}^n$ is *an optimal solution* of the LP

$$\begin{array}{ll} \min & f(\underline{x}) \\ s.t. & \\ & \underline{x} \in X \subseteq \mathbb{R}^n \end{array}$$

if $f(\underline{x}^*) \leq f(\underline{x}) \quad \forall\, \underline{x} \in X$.

A wide variety of decision-making problems can be formulated or approximated as linear programs (LPs).

They often involve the optimal allocation of a given set of limited resources to different activities.

General form:

$$\min \quad z = c_1 x_1 + \ldots + c_n x_n$$

$$a_{11} x_1 + \ \ldots + a_{1n} x_n \ \geq b_1 \quad (=) \quad (\leq)$$

$$\vdots \qquad\qquad \vdots$$

$$a_{m1} x_1 + \ldots + a_{mn} x_n \ \geq b_m \quad (=) \quad (\leq)$$

$$x_1, \ldots, x_n \geq 0$$

Matrix notation:

$$\min \quad z = \underline{c}^T \underline{x}$$

$$A\underline{x} \geq \underline{b}$$

$$\underline{x} \geq \underline{0}$$

$$\min \quad \begin{bmatrix} c_1 \ \ldots \ c_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \geq \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \geq \underline{0}$$

Historical sketch

Jean Baptiste Joseph Fourier
(1786-1830)

L.V. Kantorovitch
(1912-1986)

G.B. Dantzig
(1914-2005)

1825/26: Fourier presents a method to solve systems of linear inequalities and discusses LPs with 2-3 variables.

1939: Kantorovitch lays the foundations of **LP** (Nobel prize, 1975)

1947: Dantzig independently proposes **LP** and invents the **Simplex algorithm**.

Example 1: Diet problem

Given

n aliments $j = 1,\ldots, n$

m nutrients (basic substances) $i = 1,\ldots, m$

a_{ij} amount of i-th nutrient contained in one unit of the j-th aliment

b_i daily requirement of the i-th nutrient

c_j cost of a unit of j-th aliment,

determine a diet that minimizes the total cost while satisfying all the daily requirements.

Decision variables:

x_j = amount of j-th aliment in the diet, with $j = 1,\dots, n$

$$\min \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \qquad \forall\ i = 1, \dots, m$$

$$x_j \geq 0 \qquad \forall\ j = 1, \dots, n$$

Example 2: Transportation problem (single product)

Given

m	production plants	$i = 1,\ldots, m$
n	clients	$j = 1,\ldots, n$
c_{ij}	unit transportation cost from plant i to client j	
p_i	maximum supply (production capacity) of plant i	
d_j	demand of client j	
q_{ij}	maximum amount transportable from plant i to client j,	

determine a transportation plan that minimizes the total costs while respecting plant capacities and client demands.

Assumption: $\sum_{i=1}^{m} p_i \geq \sum_{j=1}^{n} d_j$

Decision variables: x_{ij} = amount of product transported from i to j, with $i = 1,\ldots, m$ and $j = 1,\ldots, n$.

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\sum_{j=1}^{n} x_{ij} \leq p_i \qquad \forall\, i = 1,\ldots, m \qquad \text{(plant capacity)}$$

$$\sum_{i=1}^{m} x_{ij} \geq d_j \qquad \forall\, j = 1,\ldots, n \qquad \text{(client demand)}$$

$$0 \leq x_{ij} \leq q_{ij} \qquad \forall\, i, j \qquad \text{(transportation capacity)}$$

Example 3: Production planning problem

Given

n products ($j = 1,\dots, n$) which compete for resources

m resources ($i = 1,\dots, m$)

c_j profit (selling price – cost) per unit of j-th product

a_{ij} amount of i-th resource needed to produce one unit of j-th product

b_i maximum available amount of i-th resource,

determine a production plan that maximizes the total profit given the available resources.

Decision variables:

x_j = amount of j-th product, with $j = 1,..., n$

$$\max \sum_{j=1}^{n} c_j x_j$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad \forall\ i = 1, \ldots, m$$

$$x_j \geq 0 \qquad \forall\ j = 1, \ldots, n$$

Assumptions of LP models

1) Linearity (proportionality and additivity) of the objective function and constraints.

Proportionality:

contribution of each variable = constant $\times$ variable

Drawback: does not account for economies of scale.

Additivity:

contribution of all variables = sum of single contributions

Drawback: competing products $\Rightarrow$ profits are not independent.

2) Divisibility

The variables can take fractional (rational) values.

3) Parameters are considered as constants which can be estimated with a sufficient degree of accuracy.

Uncertainty in the parameters may require more complex mathematical programs.

In Linear Programming, if we have different “scenarios” we adopt “sensitivity analysis” (see end of Chapter 4).

4.1 Equivalent forms

General form:

$$
\begin{array}{lll}
\min & z = \underline{c}^T \underline{x} & \\
(\max) & & \\
& A_1 \underline{x} \geq \underline{b}_1 & \\
& A_2 \underline{x} \leq \underline{b}_2 & \\
& A_3 \underline{x} = \underline{b}_3 & \\
& x_j \geq 0 & \text{for } j \in J \subseteq \{1,\ldots, n\} \\
& x_j \text{ free} & \text{for } j \in \{1,\ldots, n\} \setminus J
\end{array}
$$

inequality constraints

equality constraints

Definition: *Standard form*

$$
\begin{array}{ll}
\min & z = \underline{c}^T \underline{x} \\
& A\underline{x} = \underline{b} \\
& \underline{x} \geq \underline{0}
\end{array}
$$

only equality constraints and all variables non negative.

The two forms are equivalent.

Simple transformation rules allow to pass form one form to the other form.

Warning**:** the transformation may involve adding/deleting variables and/or constraints.

Transformation rules

- $\max \underline{c}^T \underline{x} = - \min - \underline{c}^T \underline{x}$

- $\underline{a}^T \underline{x} \leq b \quad \Rightarrow \quad \begin{cases} \underline{a}^T \underline{x} + s = b \\ s \geq 0 \end{cases}$ *slack* variable

- $\underline{a}^T \underline{x} \geq b \quad \Rightarrow \quad \begin{cases} \underline{a}^T \underline{x} - s = b \\ s \geq 0 \end{cases}$ *surplus* variable

- x_j unrestricted in sign $\Rightarrow$ $\begin{cases} x_j = x_j^+ - x_j^- \\ x_j^+ \geq 0 \\ x_j^- \geq 0 \end{cases}$

After substituting x_j with $x_j^+ - x_j^-$, we delete x_j from the problem.

Example

General from:

$$
\begin{array}{ll}
\max & 2x_1 - 3x_2 \\
s.t. & 4x_1 - 7x_2 \leq 5 \\
 & 6x_1 - 2x_2 \geq 4 \\
 & x_1 \geq 0, \quad x_2 \text{ unrestricted}
\end{array}
$$

$\Rightarrow$

Step 1: $x_2 = x_3 - x_4$
$x_3, x_4 \geq 0$

$$
\begin{array}{ll}
\max & 2x_1 - 3x_3 + 3x_4 \\
s.t. & 4x_1 - 7x_3 + 7x_4 \leq 5 \\
 & 6x_1 - 2x_3 + 2x_4 \geq 4 \\
 & x_1, x_3, x_4 \geq 0
\end{array}
$$

$$\Rightarrow \qquad \min \quad -2x_1 + 3x_3 - 3x_4$$

$$4x_1 - 7x_3 + 7x_4 + x_5 = 5$$

$$6x_1 - 2x_3 + 2x_4 - x_6 = 4$$

$$x_1, x_3, x_4, x_5, x_6 \geq 0$$

Step 2: introduce slack and surplus variables x_5 and x_6

Step 3: change the objective function sign

Other straightforward transformations

$$\underline{a}\,\underline{x} \le \underline{b} \;\rightarrow\; -\underline{a}\,\underline{x} \ge -\underline{b}$$

$$\underline{a}\,\underline{x} \ge \underline{b} \;\rightarrow\; -\underline{a}\,\underline{x} \le -\underline{b}$$

$$\underline{a}\,\underline{x} = \underline{b} \;\rightarrow\; \begin{cases} \underline{a}\,\underline{x} \ge \underline{b} \\ \underline{a}\,\underline{x} \le \underline{b} \end{cases} \;\rightarrow\; \begin{cases} \underline{a}\,\underline{x} \ge \underline{b} \\ -\underline{a}\,\underline{x} \ge -\underline{b} \end{cases}$$

Example

Capital budgeting

Capital of 10.000 € and two possible investments A and B with, respectively, 4% and 6% expected return.

Determine a portfolio that maximizes the total expected return, while respecting the diversification constraints:

- at most 75% of the capital is invested in A,
- at most 50% of the capital is invested in B.

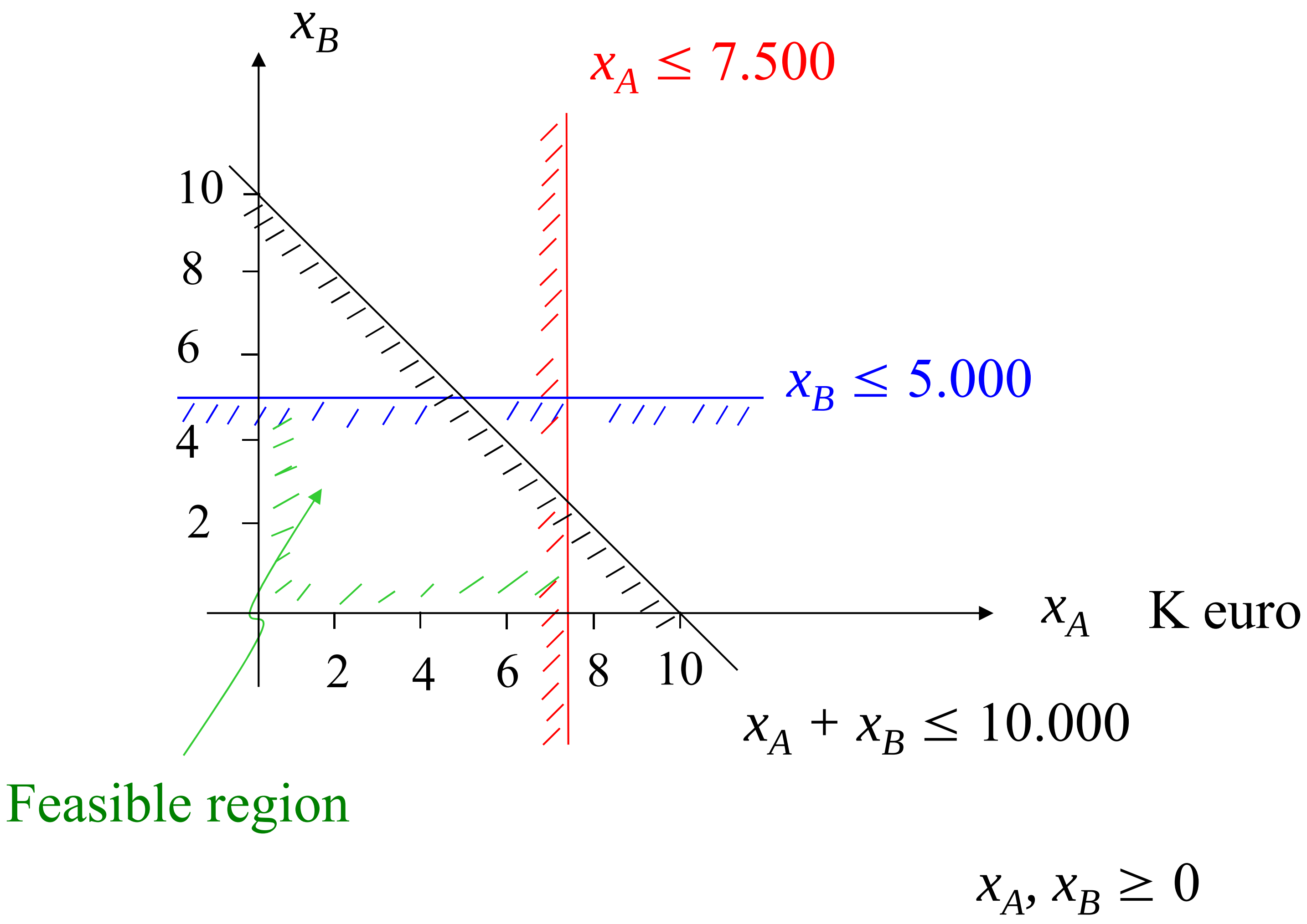
Model:

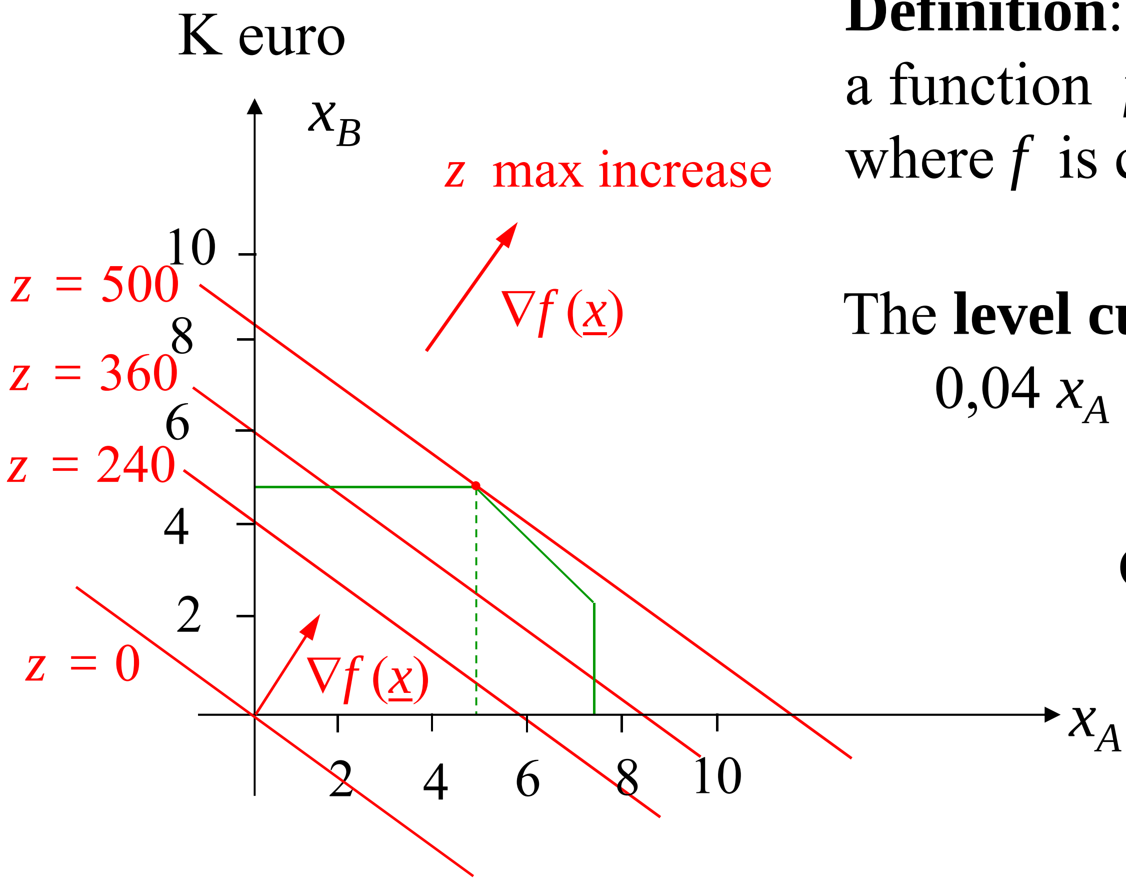
x_A = amount invested in A

x_B = amount invested in B

$$
\begin{array}{ll}
\max & z = 0{,}04\, x_A + 0{,}06\, x_B \\
s.t. & \\
& x_A + x_B \leq 10.000 \\
& x_A \leq 0.75 \cdot 10.000 \\
& x_B \leq 0.50 \cdot 10.000 \\
& x_A, x_B \geq 0
\end{array}
$$

4.2.1 Graphical solution





Definition: A *level curve of value z* of a function f is the set of points in $\mathbb{R}^n$ where f is constant and takes value z.

The **level curves** of a LP are **lines:**

$$0{,}04\, x_A + 0{,}06\, x_B = z \quad \leftarrow \text{constant}$$

Optimal solution:

$$\begin{pmatrix} x^*_A \\ x^*_B \end{pmatrix} = \begin{pmatrix} 5000 \\ 5000 \end{pmatrix}$$

$z^* = 500$

$$\nabla f(\underline{x}) = \begin{pmatrix} 0{,}04 \\ 0{,}06 \end{pmatrix}$$ is the direction at $\underline{x}$ of fastest increase of f.

4.2.2 Vertices of the feasible region

Consider a LP with inequality constraints (easier to visualize).

Definitions: $H = \{ \underline{x} \in \mathbb{R}^n : \underline{a}^T\underline{x} = b \}$ is a *hyperplane* and

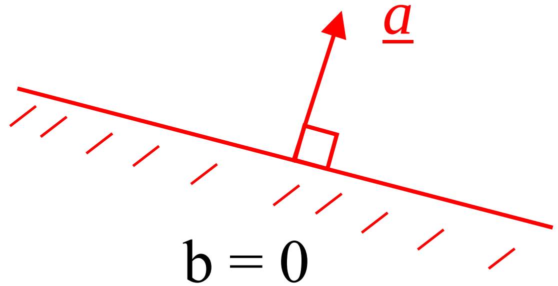
$H^- = \{ \underline{x} \in \mathbb{R}^n : \underline{a}^T\underline{x} \leq b \}$ is an *affine half-space*.

Half-plane in $\mathbb{R}^2$

Each inequality constraint ($\underline{a}^T\underline{x} \leq b$) defines an affine half-space in the variable space.

$$H^- = \{ \underline{x} \in \mathbb{R}^n : \underline{a}^T\underline{x} \leq b \}$$

$$\underline{a} \neq \underline{0}$$

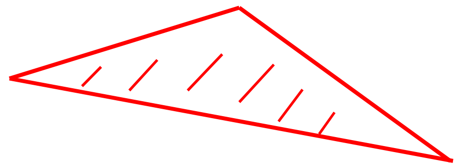


b = 0

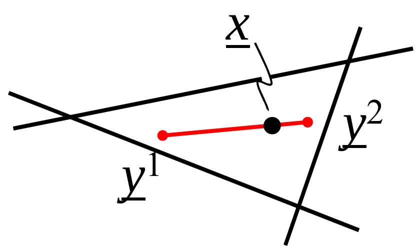
Definition: The feasible region X is a *polyhedron P*

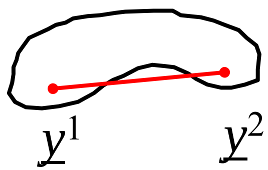
$\cap$ half-spaces
\# finite

P can be empty or unbounded



Definition: A subset X $\subseteq \mathbb{R}^n$ is *convex* if for each pair $\underline{y}^1, \underline{y}^2 \in$ X, X contains the whole segment connecting $\underline{y}^1$ and $\underline{y}^2$.



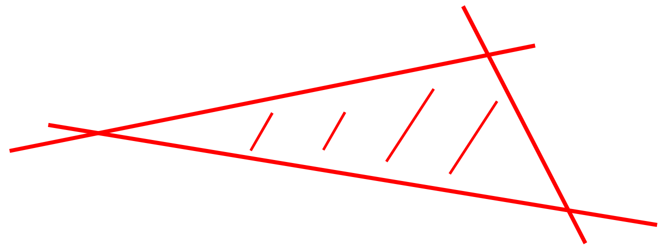


$\neq$ convex

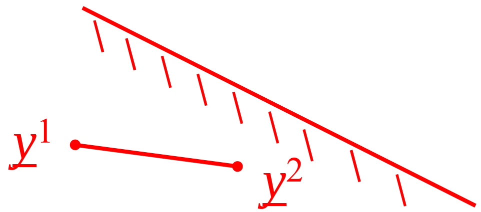
$$[\underline{y}^1, \underline{y}^2] = \{\ \underline{x} \in \mathbb{R}^n : \underline{x} = \alpha\, \underline{y}^1 + (1 - \alpha)\, \underline{y}^2 \quad \text{with} \quad \alpha \in [0, 1]\ \}$$

segment $\equiv$ {all the convex combinations of $\underline{y}^1$ e $\underline{y}^2$ }

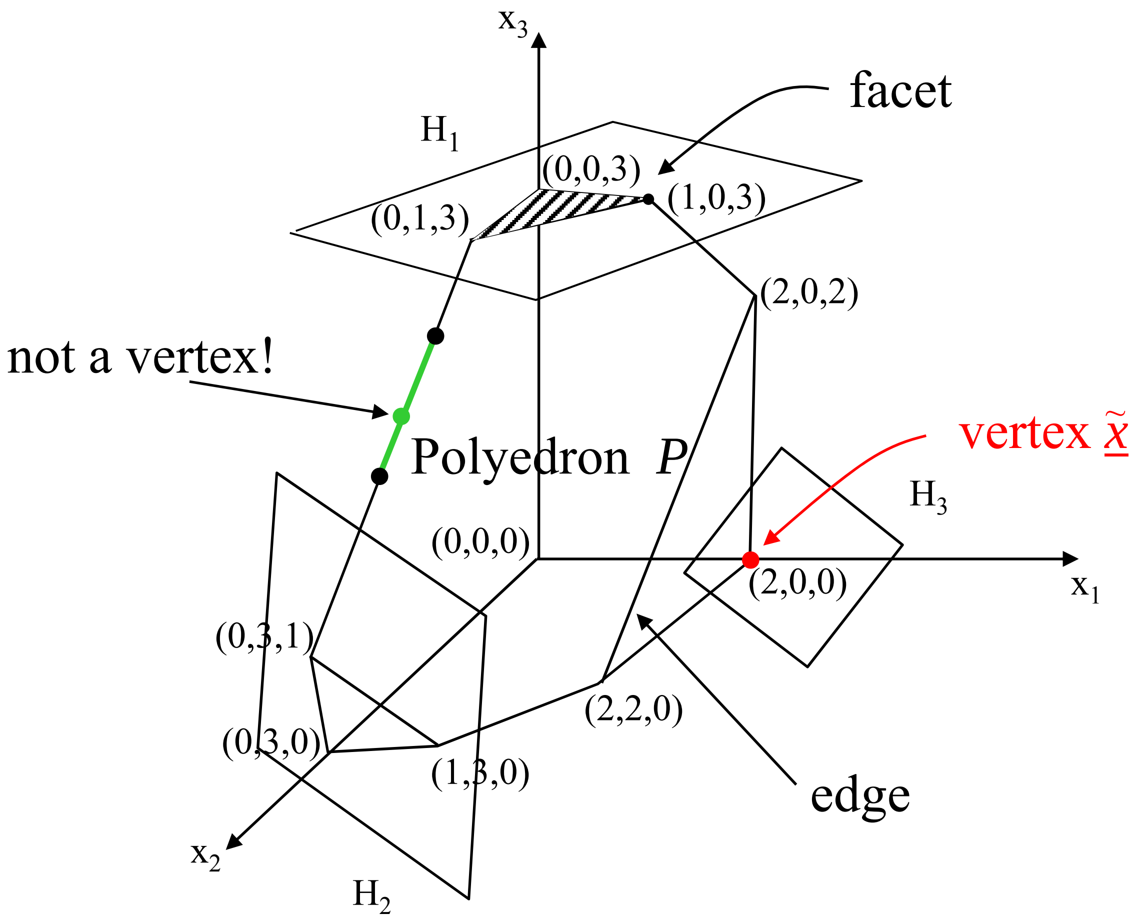
Property: A polyhedron P is a convex set of $\mathbb{R}^n$.



Indeed: any half-space is clearly convex



and the intersection of a finite number of convex sets is also a convex set.



Algebraically:

$\underline{\tilde{x}}$ is a vertex of P

if $\nexists\ \underline{y}^1, \underline{y}^2 \in P$, $\underline{y}^1 \neq \underline{y}^2$

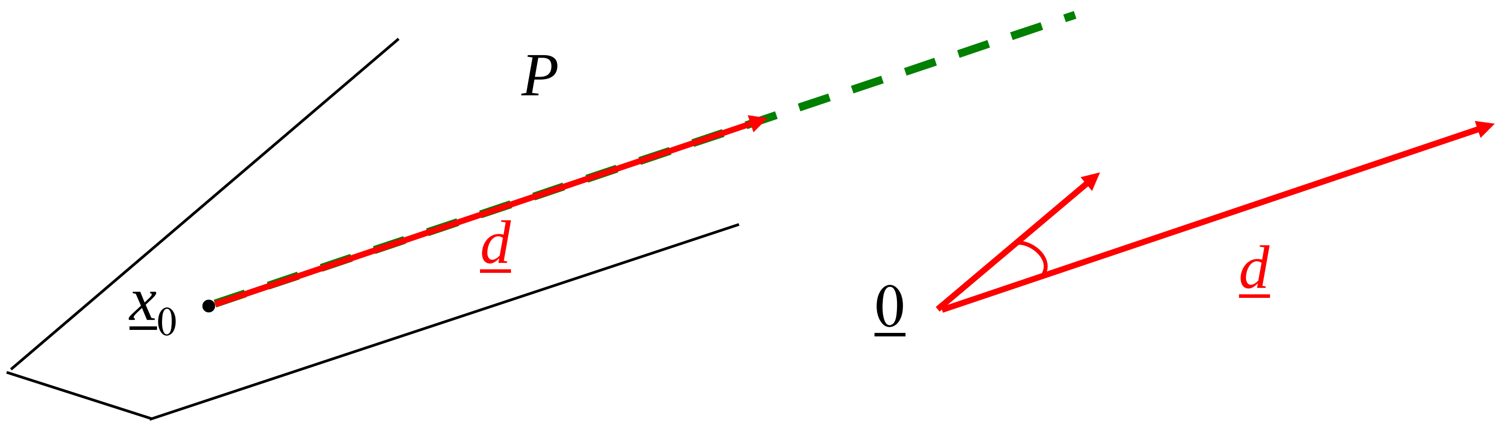
and $\alpha \in (0, 1)$ s.t.

$\underline{\tilde{x}} = \alpha\, \underline{y}^1 + (1 - \alpha)\, \underline{y}^2$

Definition: A *vertex* of P is a point of P which cannot be expressed as a convex combination of two other distinct points of P.

Property: A non-empty polyhedron $P = \{ \underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{b}, \underline{x} \geq \underline{0} \}$ (in standard form) or $P = \{ \underline{x} \in \mathbb{R}^n : A\underline{x} \geq \underline{b}, \underline{x} \geq \underline{0} \}$ (in canonical form) has a finite number (≥ 1) of vertices.

Definition: Given a polyhedron P, a vector $\underline{d} \in \mathbb{R}^n$ with $\underline{d} \neq \underline{0}$ is an *unbounded feasible direction* of P if, for every point $\underline{x}_0 \in P$, the "ray" $\{ \underline{x} \in \mathbb{R}^n : \underline{x} = \underline{x}_0 + \lambda \, \underline{d} \, , \lambda \geq 0 \}$ is contained in P.



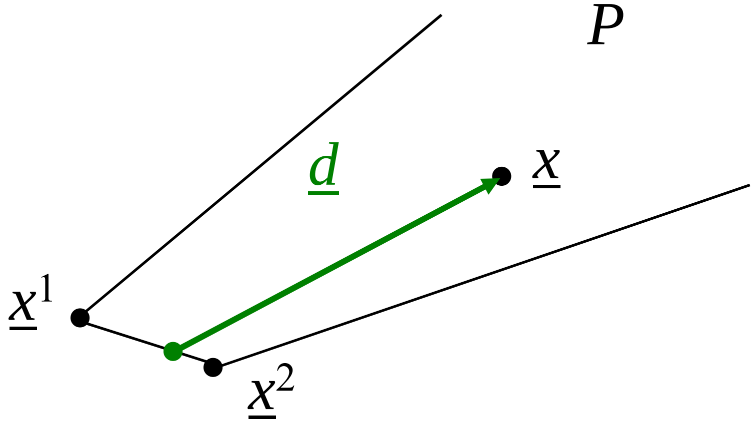
Theorem (representation of polyhedra -- Weyl-Minkowski):

Every point $\underline{x}$ of a polyhedron P can be expressed as a convex combination of its vertices $\underline{x}^1,\ldots,\underline{x}^k$ plus (if needed) an unbounded feasible direction $\underline{d}$ of P :

$$\underline{x} = \alpha_1 \underline{x}^1 + \ldots + \alpha_k \underline{x}^k + \underline{d}$$

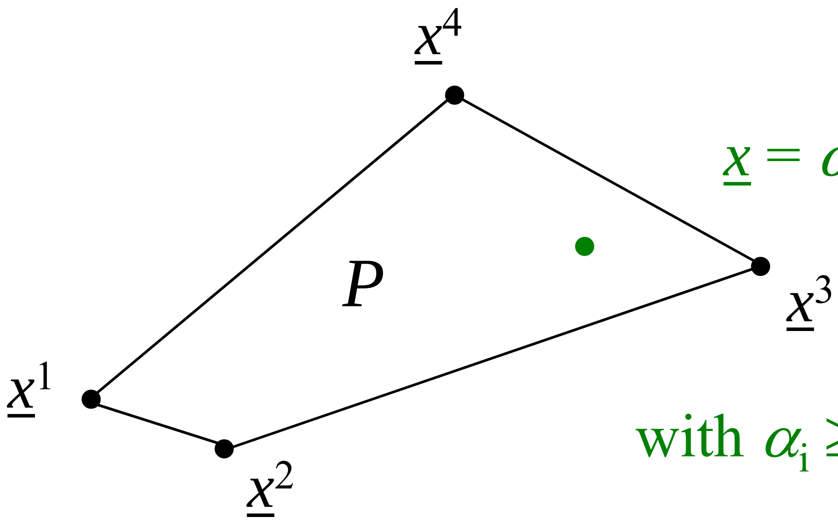
where the multipliers $\alpha_i \geq 0$ satisfy $\alpha_1 + \ldots + \alpha_k = 1$.

$\forall \underline{x} \in [\underline{x}^1, \underline{x}^2], \ \underline{d} = \underline{0}$



Definition: A *polytope* is a bounded polyhedron, that is, it has the only unbounded feasible direction $\underline{d} = \underline{0}$.

Consequence: Every point $\underline{x}$ of a polytope P can be expressed as a convex combination of its vertices.



$\underline{x} = \alpha_1 \underline{x}^1 + \ldots + \alpha_4 \underline{x}^4$ $\quad (\underline{d} = \underline{0})$

with $\alpha_i \geq 0$ and $\alpha_1 + \ldots + \alpha_4 = 1$

Fundamental theorem of Linear Programming:

Consider a LP $\min\{ \underline{c}^T\underline{x} : \underline{x} \in P \}$ where $P \subseteq \mathbb{R}^n$ is a non-empty polyhedron of the feasible solutions (in standard or canonical form). Then either there exists (at least) one optimal vertex or the value of the objective function is unbounded below on P.

Proof

Case 1: P has an unbounded feasible direction $\underline{d}$ such that $\underline{c}^T\underline{d} < 0$

P is unbounded and the values $z = \underline{c}^T\underline{x} \rightarrow -\infty$

along the direction $\underline{d}$

Case 2: P has no unbounded feasible direction $\underline{d}$ such that $\underline{c}^T\underline{d} < 0$, that is, for all of them we have $\underline{c}^T\underline{d} \geq 0$.

Any point of P can be expressed as:
$$\underline{x} = \sum_{i=1}^{k} \alpha_i \, \underline{x}^i + \underline{d}$$

where $\underline{x}^1, \ldots, \underline{x}^k$ are the vertices of P, $\alpha_i \geq 0$ with $\alpha_1 + \ldots + \alpha_k = 1$, and $\underline{d} = \underline{0}$, or $\underline{d}$ is a unbounded feasible direction.

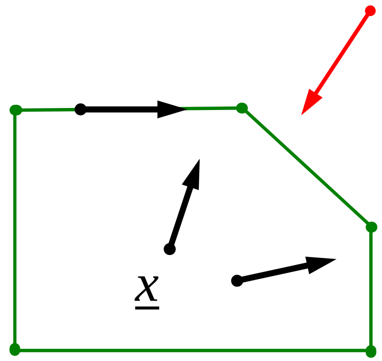
For any $\underline{x} \in P$, we have $\underline{d} = \underline{0}$ or $\underline{c}^T\underline{d} \geq 0$ and hence

$$\underline{c}^T \underline{x} = \underline{c}^T \left(\sum_{i=1}^{k} \alpha_i \, \underline{x}^i + \underline{d} \right) = \sum_{i=1}^{k} \alpha_i \, \underline{c}^T \underline{x}^i + \underbrace{\underline{c}^T \underline{d}}_{\substack{\geq \\ 0}} \geq \min_{1 \leq i \leq k} \left\{ \underline{c}^T \underline{x}^i \right\}$$

since $\alpha_i \geq 0 \;\; \forall i$ and $\alpha_1 + \ldots + \alpha_k = 1$.

Geometrically

An interior point $x \in P$ cannot be an optimal solution:

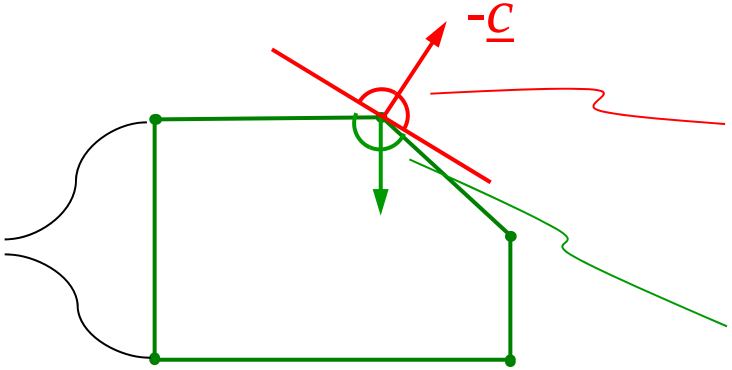


$\underline{c}$ = direction of fastest increase in z (constant gradient)

$\exists$ always an improving direction

In an optimal vertex all feasible directions (respecting feasibility for a sufficiently small step) are "worsening" directions:

vertices



improving directions

feasible directions

The theorem implies that, although the variables can take fractional values, Linear Programs can be viewed as combinatorial problems:

we "only" need to examine the vertices of the polyhedron of the feasible solutions!

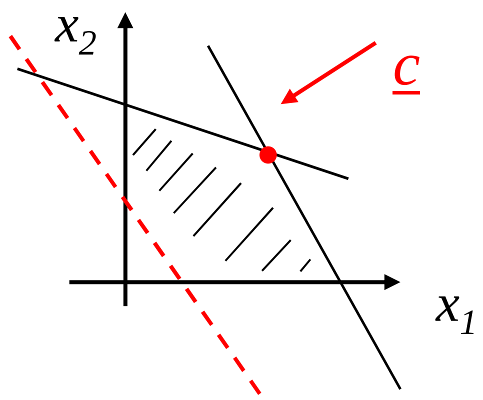
Finite but often exponential number

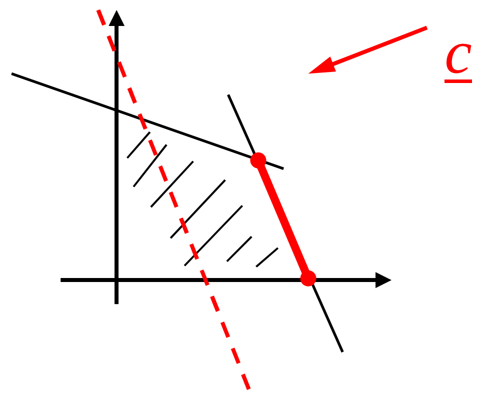
Graphical method only applicabile for $n \leq 3$.

4.2.3 Four types of Linear Programs

Observation: Since min $\underline{c}^T\underline{x}$, better solutions found by moving along -$\underline{c}$.

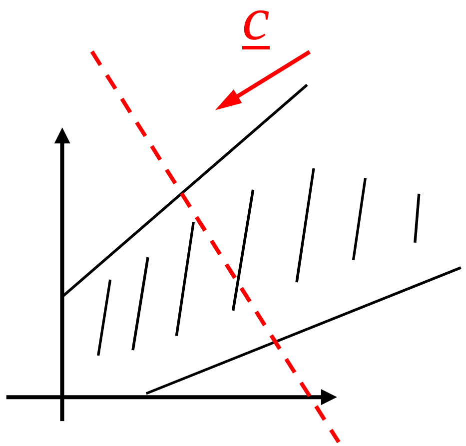
- A unique optimal solution





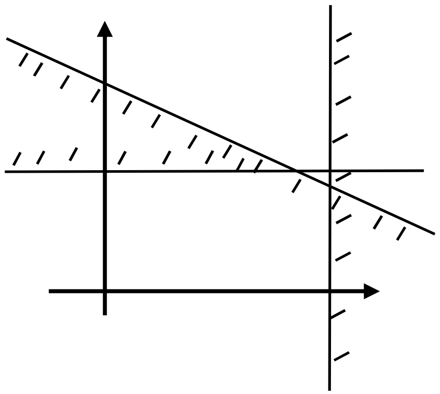
- Multiple (infinitely many) optimal solutions

- Unbounded LP



Unbounded polyhedron and unlimited objective function value

- Infeasible LP



Empty polyhedron (no feasible solution)