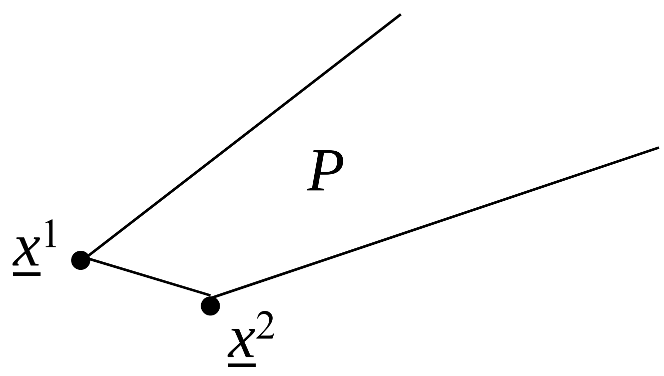
4.3 Basic feasible solutions and vertices of polyhedra

Due to the fundamental theorem of Linear Programming, to solve any LP it ‘suffices’ to consider the vertices (finitely many) of the polyhedron P of the feasible solutions.

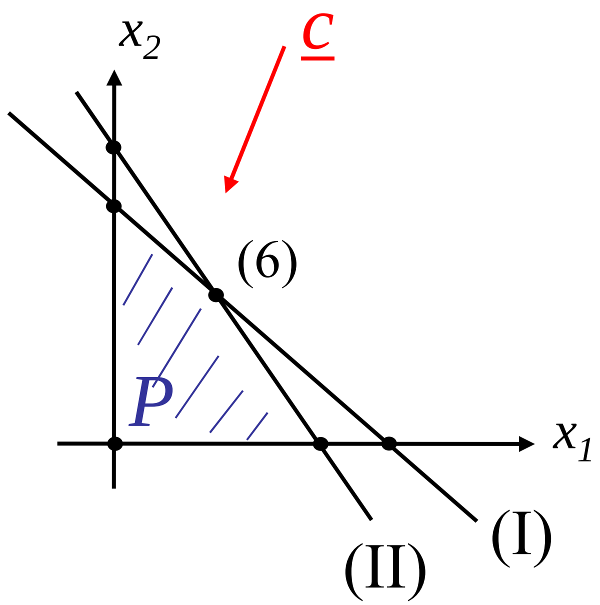


Since the geometrical definition of vertex cannot be exploited algorithmically, we need an algebraic characterization.

Which are the vertices of $P = \{ \underline{x} \in \mathrm{R}^n : A\underline{x} \leq \underline{b}, \underline{x} \geq 0\}$ with only inequalities?

Example:

$$\begin{aligned} \min\ & -x_1 - 3x_2 \\ \text{s.t.}\ & x_1 + x_2 \leq 6 \quad \text{(I)} \\ & 2x_1 + x_2 \leq 8 \quad \text{(II)} \\ & x_1, x_2 \geq 0 \end{aligned}$$



Vertex corresponds to the intersection of the hyperplanes associated to n inequalities.

Example: $n=2$

Vertex (6) is the intersection of hyperplanes of (I) and (II), i.e., solution of equations $x_1 + x_2 = 6$ and $2x_1 + x_2 = 8$.

What about the vertices of polyhedra expressed in standard form?

$$P = \{\underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{b}, \underline{x} \geq 0\}$$

We want to solve LPs in standard form

Easier to describe if we start from a polyhedron

$$P = \{\underline{x} \in \mathbb{R}^n : A\underline{x} \leq \underline{b}, \underline{x} \geq 0\},$$

transform it into standard form

$$P' = \{\underline{x} \in \mathbb{R}^n : A\underline{x} + \underline{s} = \underline{b}, \underline{x}, \underline{s} \geq 0\}$$

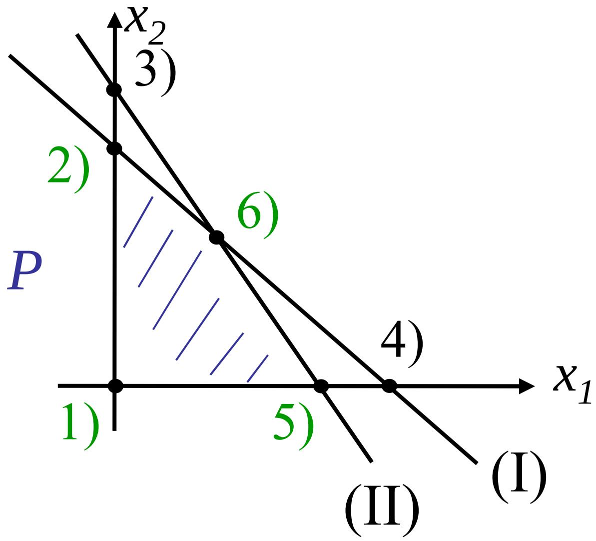
and rename: A:= [A| I], $\underline{x}$:= [$\underline{x}$ | $\underline{s}$].

Example:

(*P*) $x_1 + x_2 \leq 6$ (I)
$2x_1 + x_2 \leq 8$ (II)
$x_1, x_2 \geq 0$

$\Longrightarrow$

(*P'*) $x_1 + x_2 + s_1 = 6$
$2x_1 + x_2 + s_2 = 8$
$x_1, x_2, s_1, s_2 \geq 0$



Taking the intersection of the lines associated to (I) and (II) in *P*, amounts in *P'* to let $s_1 = s_2 = 0$.

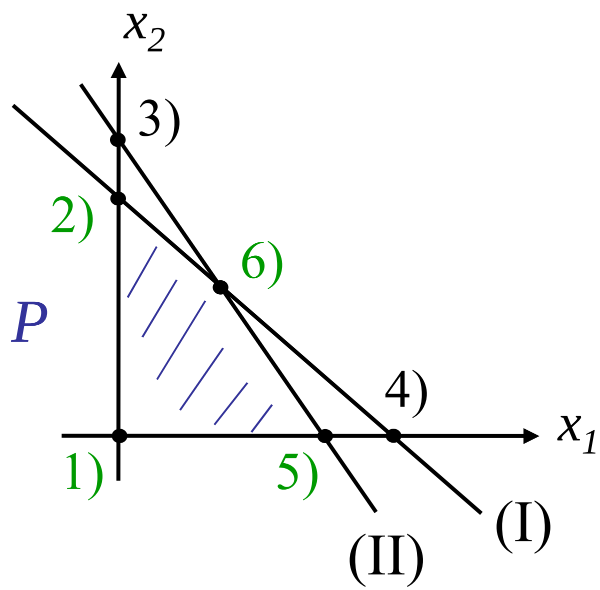
Observation: Every constraint in *P* corresponds to a slack variable in *P*', when the slack variable is set to 0 the constraint is satisfied with =.

Example: $s_1 \leftrightarrow x_1 + x_2 \leq 6,\ x_1 \leftrightarrow x_1 \geq 0$

Vertex of *P* is the intersection of *n* inequalities in *P'* it is equivalent to set the corresponding variables in *P'* to 0.

Example (continued): Compute all the intersections

$$x_1 + x_2 + s_1 = 6 \quad \text{(I)}$$
$$2x_1 + x_2 + s_2 = 8 \quad \text{(II)}$$
$$x_1, x_2, s_1, s_2 \geq 0$$



1) $x_1 = 0, x_2 = 0 \Rightarrow s_1 = 6, \; s_2 = 8$

2) $x_1 = 0, s_1 = 0 \Rightarrow x_2 = 6, \; s_2 = 2$

3) $x_1 = 0, s_2 = 0 \Rightarrow x_2 = 8, \; s_1 = -2$

4) $x_2 = 0, s_1 = 0 \Rightarrow x_1 = 6, \; s_2 = -4$

5) $x_2 = 0, s_2 = 0 \Rightarrow x_1 = 4, \; s_1 = 2$

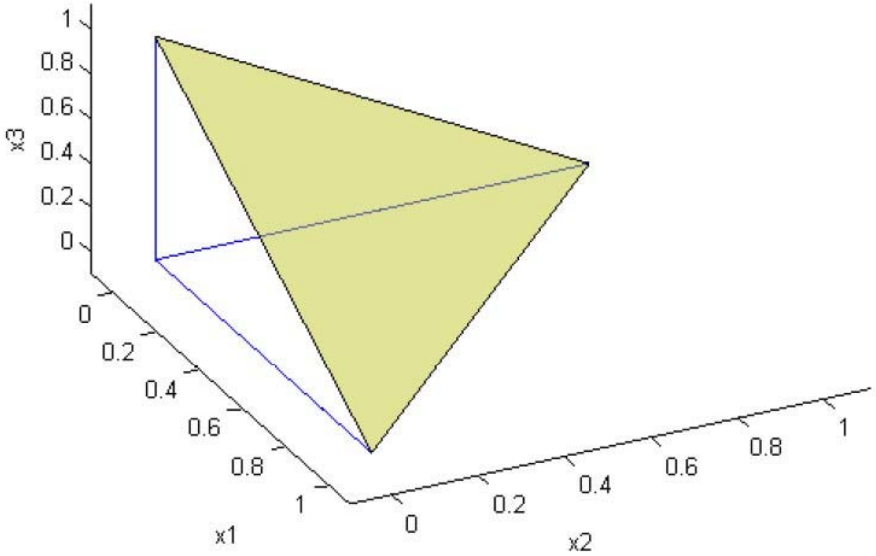
6) $s_1 = 0, \quad s_2 = 0 \Rightarrow x_1 = 2, \; x_2 = 4$

The intersections where some x_j or s_i are < 0 yield infeasible solutions.

Which are the vertices of a polyhedron in standard form?

Example:

$P=\{\underline{x} \in \mathbb{R}^3: x_1+x_2+x_3=1, \underline{x} \geq 0\}$



Property: For any polyhedron $P =\{ \underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{b}, \underline{x} \geq 0\}$

- the facets (edges in $\mathbb{R}^2$) are obtained by setting one variable to 0,
- the vertices are obtained by setting *n-m* variables to 0.

In the example: 3-1=2 variables set to 0 for vertices.

Algebraic characterization of the vertices

Consider any $P = \{\underline{x} \in \mathrm{R}^n : A\underline{x} = \underline{b}, \underline{x} \geq 0\}$ in standard form.

Assumption: $A \in \mathbb{R}^{m \times n}$ with $\boldsymbol{m \leq n}$ of rank m (A is of full rank)

Equivalent to assume that there are no “redundant” constraints.

Example:

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 2 \quad \text{(I)} \\ x_1 + x_2 &= 1 \quad \text{(II)} \\ x_1 + x_3 &= 1 \quad \text{(III)} \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Since (I) = (II) + (III), then (I) can be dropped.

If $m = n$, $\exists$ unique solution of $A\underline{x} = \underline{b}$. ($\underline{x} = A^{-1}\underline{b}$)

If $m < n$, $\exists\ \infty$ solution of $A\underline{x} = \underline{b}$: the system has n-m degrees of freedom (n-m variables can be fixed arbitrarily). If we fix them to 0, we get a vertex.

$$P = \{\underline{x} \in \mathrm{R}^n : A\underline{x} = \underline{b}, \underline{x} \geq 0\}$$

n variables, m constraints, A in $\mathrm{R}^{m \times n}$

Definition: A *basis* of such a matrix A is a subset of m columns of A that are linearly independent and form an $m \times m$ non singular matrix B.

$$A = \Big[\underbrace{B}_{m} \,\vdots\, \underbrace{N}_{n-m}\Big]$$

First permute the columns of A, then partition A into $[B|N]$

Let $\underline{x} = \begin{pmatrix} \underline{x}_B \\ ---- \\ \underline{x}_N \end{pmatrix}$ $\begin{matrix} \} \; m \text{ components} \\ \} \; n\text{-}m \text{ components} \end{matrix}$

Any system $A\underline{x} = \underline{b}$ can be written as

$$B\,\underline{x}_B + N\,\underline{x}_N = \underline{b}$$

B is nonsigular

For any set of values for $\underline{x}_N$, we have

$$\underline{x}_B = B^{-1}\underline{b} - B^{-1}N\,\underline{x}_N$$

Definitions:

- A ***basic solution*** is a solution obtained by setting $\underline{x}_N = \underline{0}$ and, consequently, letting $\underline{x}_B = B^{-1}\underline{b}$.
- A basic solution with $\underline{x}_B \geq 0$ is a ***basic feasible solution***.
- The variables in $\underline{x}_B$ are the *basic variables* and those in $\underline{x}_N$ are the ***non basic variables***.

Note: $\underline{x}_B$ satisfies $A\underline{x} = \underline{b}$ by construction.

Theorem: $\underline{x} \in \mathbb{R}^n$ is a basic feasible solution

if and only if

$\underline{x}$ is a vertex of $P = \{ \underline{x} \in \mathbb{R}^n : A\underline{x} = \underline{b}, \underline{x} \geq \underline{0} \}$.

Example:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 4 \\ 2 \\ 3 \\ 6 \end{pmatrix}$$

$$\begin{aligned} \min z &= 2x_1 + x_2 + 5x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 = 4 \\ & x_1 + x_5 = 2 \\ & x_3 + x_6 = 3 \\ & 3x_2 + x_3 + x_7 = 6 \\ & x_i \geq 0 \quad i = 1, \ldots, 7 \end{aligned}$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Choosing columns 4, 5, 6, 7, we have:

$$B^{-1} = B \Rightarrow \underline{x}_B = B^{-1}\underline{b} = \underline{b} \geq \underline{0}$$

basic feasible solution

Example (continued):

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 4 \\ 2 \\ 3 \\ 6 \end{pmatrix}$$

$$\min z = 2x_1 + x_2 + 5x_3$$

s.t.

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 4 \\ x_1 + x_5 &= 2 \\ x_3 + x_6 &= 3 \\ 3x_2 + x_3 + x_7 &= 6 \\ x_i \geq 0 \quad i &= 1, \ldots, 7 \end{aligned}$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix}$$

Choosing columns 2, 5, 6, 7, we have:

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \quad \underline{x}_B = \begin{pmatrix} 4 \\ 2 \\ 3 \\ -6 \end{pmatrix}$$

infeasible!

Number of basic feasible solutions

At most one for each choice of *n*-*m* variables out of *n* (nonbasic variables)

$$\#\text{ basic feasible solutions} \leq \binom{n}{n-m} = \frac{n!}{(n-m)!\,(n-(n-m))!} = \binom{n}{m}$$