4.5 Simplex method

LP in standard form:

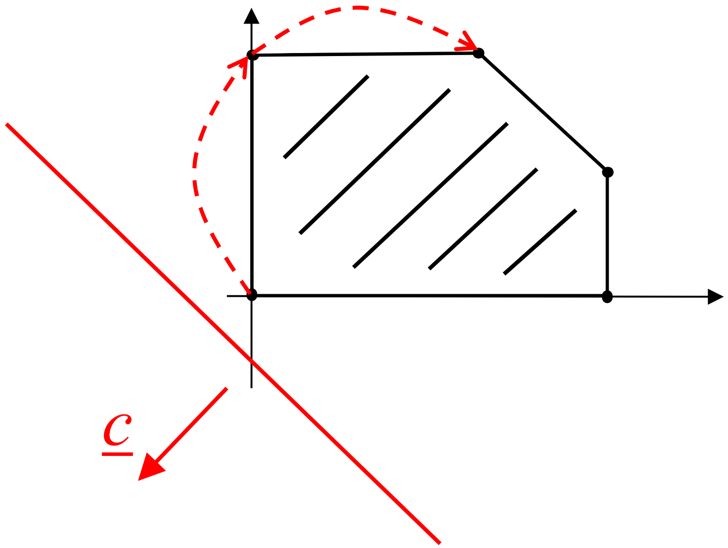
$$\begin{aligned} \min \quad & z = \underline{c}^T \underline{x} \\ s.t. \quad & \\ & A\underline{x} = \underline{b} \\ & \underline{x} \geq \underline{0} \end{aligned}$$

George Dantzig (1914-2005)

Examine a sequence of basic feasible solutions with non increasing objective function values until an optimal solution is reached or the LP is found to be unbounded (G. Dantzig 1947).

At each iteration, we move from a basic feasible solution to a "neighboring" basic feasible solution.

Geometrically:



Generate a path along the edges of the polyhedron of the feasible solutions until an optimal vertex is reached.

a sequence of adjacent vertices

Given the correspondence between the basic feasible solutions and the vertices, we need to describe how to :

- Find an initial vertex or establish that the LP is infeasible. By applying the same method to another LP, see end of chapter.

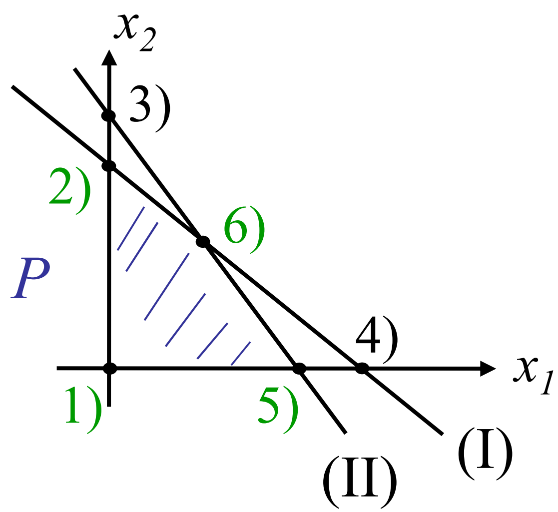
- Move from a current vertex to a better adjacent vertex (in terms of objective function value) or establish that the LP is unbounded.

- Determine whether the current vertex is optimal.

4.5.1 Move to an adjacent vertex

Example:

$$x_1 + x_2 + s_1 = 6 \quad \text{(I)}$$
$$2x_1 + x_2 + s_2 = 8 \quad \text{(II)}$$
$$x_1, x_2, s_1, s_2 \geq 0$$



Move from vertex 1) to vertex 5):

In 1) $x_1 = 0, x_2 = 0 \Rightarrow s_1 = 6,\ s_2 = 8$ with $x_B=(s_1,s_2)$ and $x_N=(x_1,x_2)$

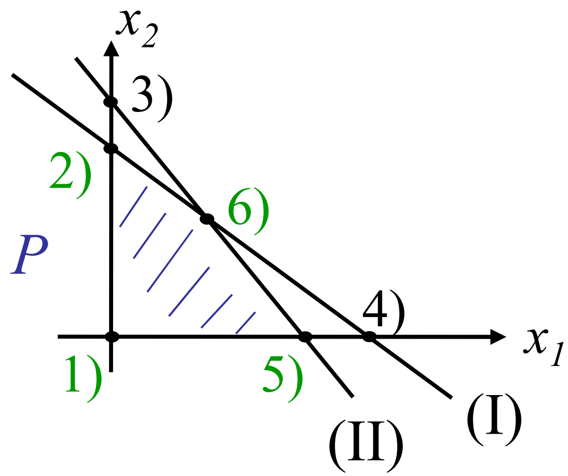
In 5) $x_2 = 0, s_2 = 0 \Rightarrow x_1 = 4,\ s_1 = 2$ with $x_B=(x_1,s_1)$ and $x_N=(x_2,s_2)$

Thus x_1 enters the basis B and s_2 exits the basis B.

Observation: When moving from the current vertex to an adjacent vertex, we substitute one column of B (that of s_2) with one column of N (that of x_1).

By expressing the basic variables in terms of the non basic variables, we obtain

$$s_1 = 6 - x_1 \quad - x_2$$
$$s_2 = 8 - 2x_1 - x_2$$



Now we increase x_1 while keeping $x_2 = 0$.

Since $s_1 = 6 - x_1 \geq 0$ implies $x_1 \leq 6$ and $s_2 = 8 - 2x_1 \geq 0$ implies $x_1 \leq 8/2=4$, the upper limit on the increase of x_1 is: $x_1 \leq \min\{6, 4\}=4$.

We move from vertex 1) to vertex 5) by letting x_1 enter the basis and s_2 exit the basis ($s_1 = 2$ and $s_2 = 0$).

Note: When $x_1 = 6$, we obtain the infeasible basic solution 4).

General case:

Given a basis B, the system

$$B\underline{x}_B + N\underline{x}_N = \underline{b}$$

$$A\underline{x} = \underline{b} \Leftrightarrow \sum_{j=1}^{n} a_{ij} x_j = b_i \text{ for } i = 1, \ldots, m$$

can be expressed in *canonical form*

$$\underline{x}_B + B^{-1} N \underline{x}_N = B^{-1}\underline{b} \Leftrightarrow \underline{x}_B + \bar{N}\underline{x}_N = \underline{\bar{b}}$$

which emphasizes the basic feasible solution $(\underline{x}_B, \underline{x}_N) = (B^{-1}\underline{b}, \underline{0})$.

This amounts to pre-multiply the system by B^{-1} :

$$\underbrace{B^{-1}B}_{\mathrm{I}}\, \underline{x}_B + \underbrace{B^{-1}N}_{\bar{\mathrm{N}}}\, \underline{x}_N = \underbrace{B^{-1}\underline{b}}_{\underline{\bar{b}}}$$

In the canonical form

$$x_{B_i} + \sum_{j=1}^{n-m} \bar{a}_{ij} x_{N_j} = \bar{b}_i \text{ for } i = 1,\ldots,m$$

$$I\,\underline{x}_B + \bar{N}\,\underline{x}_N = \underline{\bar{b}}$$

the basic variables are expressed in terms of the non basic variables:

$$\underline{x}_B = \underline{\bar{b}} - \bar{N}\,\underline{x}_N.$$

If we increase the value of a non basic x_s (from value 0) while keeping all the other non basic variables to 0, the system becomes

$$x_{B_i} + \bar{a}_{is} x_s = \bar{b}_i \Leftrightarrow x_{B_i} = \bar{b}_i - \bar{a}_{is} x_s \quad \text{for } i = 1,\ldots,m$$

To guarantee $x_{B_i} \geq 0$ for each i, we need to satisfy

$$\bar{b}_i - \bar{a}_{is} x_s \geq 0 \Leftrightarrow x_s \leq \frac{\bar{b}_i}{\bar{a}_{is}} \text{ for } \bar{a}_{is} > 0$$

The value of x_s can be increased up to

$$\theta^* = \min_{i=1,\dots,m} \left\{ \frac{\bar{b}_i}{\bar{a}_{is}} \text{ for } \bar{a}_{is} > 0 \right\}$$

If $\bar{a}_{is} \leq 0$ for every i, there is no limit to the increase of x_s

The value of the basic variable x_r of index

$$r = \arg\min_{i=1,\dots,m} \left\{ \frac{\bar{b}_i}{\bar{a}_{is}} \text{ for } \bar{a}_{is} > 0 \right\}$$

decreases to 0 and exits from the basis.

4.5.2 Reduced costs and optimality test

Given a LP $\min\{ \underline{c}^T \underline{x} : A\underline{x} = \underline{b},\ \underline{x} \geq \underline{0} \}$

and a feasible basis B of A, $A\underline{x} = \underline{b}$ can be rewritten as

$$B\,\underline{x}_B + N\,\underline{x}_N = \underline{b} \quad \Rightarrow \quad \underline{x}_B = B^{-1}\underline{b} - B^{-1}N\,\underline{x}_N$$

with $B^{-1}\underline{b} \geq \underline{0}$.

Basic feasible solution: $\underline{x}_B = B^{-1}\underline{b}$, $\underline{x}_N = \underline{0}$

By substitution we express the objective function in terms of only the non basic variables (for the current basis B):

$$\underline{c}^T\underline{x} = (\underline{c}^T{}_B\ \underline{c}^T{}_N)\begin{pmatrix}\underline{x}_B\\ \underline{x}_N\end{pmatrix} = (\underline{c}^T{}_B\ \underline{c}^T{}_N)\begin{pmatrix}B^{-1}\underline{b} - B^{-1}N\underline{x}_N\\ \underline{x}_N\end{pmatrix}$$

$$\begin{aligned}\underline{c}^T\underline{x} &= \underline{c}^T{}_B B^{-1}\underline{b} - \underline{c}^T{}_B B^{-1}N\underline{x}_N + \underline{c}^T{}_N\underline{x}_N\\ &= \underbrace{\underline{c}^T{}_B B^{-1}\underline{b}} + (\underbrace{\underline{c}^T{}_N - \underline{c}^T{}_B B^{-1}N}) \underline{x}_N\end{aligned}$$

$\Leftarrow$ only in terms of the non basic variables

z_0 = cost of the basic feasible solution

$\underline{x}_B = B^{-1}\underline{b},\ \ \underline{x}_N = \underline{0}$

$$\underline{\bar{c}}^T{}_N := \underline{c}^T{}_N - \underline{c}^T{}_B B^{-1}N$$

reduced costs of the non basic variables $\underline{x}_N$

Definition:

$$\underline{\bar{c}}^T := \underline{c}^T - \underline{c}^T_B B^{-1} A = [\overbrace{\underline{c}^T_B - \underline{c}^T_B \underbrace{B^{-1}B}_{I}}^{\underline{0}^T},\ \overbrace{\underline{c}^T_N - \underline{c}^T_B B^{-1} N}^{\underline{\bar{c}}^T_N}]$$

$\underline{\bar{c}}$ is the vector of *reduced costs* with respect to the *basis B*.

Defined for non basic as well as basic variables

$\bar{c}_j$ = change in the objective function value if the non basic variable x_j is increased by 1 unit while the other non basic variables are kept equal to 0.

The solution value changes by $\Delta z = \theta^* \bar{c}_j$

Optimality test

Given any LP $\min\{\, \underline{c}^T\underline{x} : A\underline{x} = \underline{b},\ \underline{x} \geq \underline{0} \,\}$ ($\max\{\ldots\}$) and a feasible basis B. If all reduced costs of the non basic variables $\overline{\underline{c}}_N$ are non negative (non positive) the basic feasible solution $(\underline{x}^T{}_B, \underline{x}^T{}_N)$, where $\underline{x}_B = B^{-1}\underline{b} \geq \underline{0}$ and $\underline{x}_N = \underline{0}$, of cost $\underline{c}^T{}_B B^{-1}\underline{b}$ is optimal.

Proof

$\overline{\underline{c}}^T \geq \underline{0}^T$ implies that

$$\underline{c}^T\underline{x} = \underline{c}^T{}_B B^{-1}\underline{b} + \overline{\underline{c}}^T{}_N \underline{x}_N \geq \underline{c}^T{}_B B^{-1}\underline{b} \qquad \forall\ \underline{x} \geq \underline{0},\ A\,\underline{x} = \underline{b}.$$

Observation: This optimality condition is sufficient but in general not necessary.

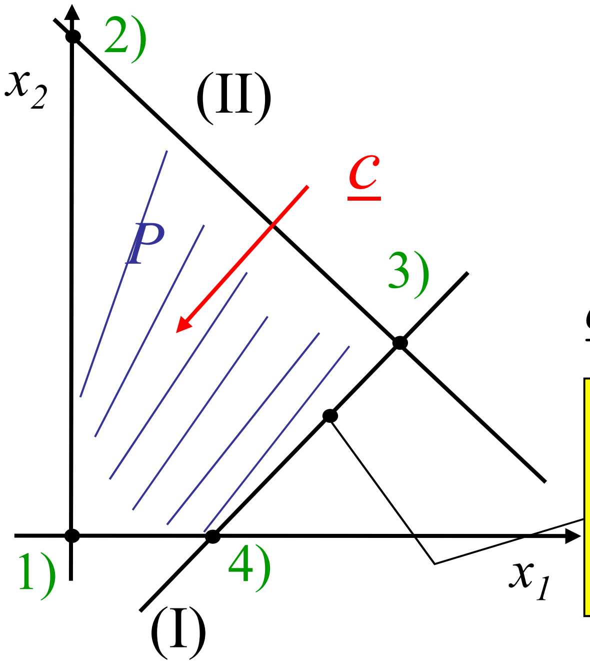
Example:

$$\min -x_1 - x_2$$

$$x_1 - x_2 + s_1 = 1 \quad \text{(I)}$$

$$x_1 + x_2 + s_2 = 3 \quad \text{(II)}$$

$$x_1, x_2, s_1, s_2 \geq 0$$



$$A = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\underline{c}^T = \begin{pmatrix} -1 & -1 & 0 & 0 \end{pmatrix}$$

Increasing x_2 from *0* to *1* keeping s_1=*0*, we proceed along the edge to (x_1,x_2) = $(2,1)$, z = *-3*

In (4): $\underline{x}_B = (\underline{x}_1,\underline{s}_2) = (1,2)$ and z = -1
$\underline{x}_N = (\underline{x}_2\ \underline{s}_1)$

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad N = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\underline{c}^T{}_B = \begin{pmatrix} -1 & 0 \end{pmatrix} \quad \underline{c}^T{}_N = \begin{pmatrix} -1 & 0 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$\overline{\underline{c}}^T{}_N = \underline{c}^T{}_N - \underline{c}^T{}_B B^{-1} N = \begin{pmatrix} -2 & 0 \end{pmatrix}$$

Since $\overline{c}_2$ = -2 < 0, increasing x_2 to 1 (keeping the other non basic variables to *0*) we improve the solution by *-2*

4.5.3 Changing basis (for minimization LP)

Consider a feasible basis B and a non basic x_s (in $\underline{x}_N$) with reduced cost $\bar{c}_s < 0$.

Increase x_s as much as possible (x_s "enters the basis") while keeping the other non basic variables equal to 0.

The basic variable x_r (in $\underline{x}_B$) such that $x_r \geq 0$ imposes the tightest upper bound θ^* on the increase of x_s (x_r leaves the basis).

If $\theta^* > 0$, the new basic feasible solution has a better objective function value.

The new basis differs w.r.t. the previous one by a single column (adjacent vertices).

To go from the canonical form of the current basic feasible solution

$$B^{-1}B\,\underline{x}_B + B^{-1}N\,\underline{x}_N = B^{-1}\underline{b}$$

to that of an adjacent basic feasible solution, it is not necessary to compute B^{-1} from scratch.

B^{-1} of the new basis B can be obtained incrementally by applying to the inverse of the previous basis (which differs w.r.t a single column) a unique "pivoting" operation.

"Pivoting" operation

Same operations used in the Gaussian elimination method to solve systems of linear equations.

Given $A\underline{x} = \underline{b}$

1. Select a coefficient $\bar{a}_{rs} \neq 0$ (the "*pivot*")
2. Divide the r-th row by $\bar{a}_{rs}$
3. For each row i with $i \neq r$ and $\bar{a}_{is} \neq 0$, substract the resulting r-th row multiplied by $\bar{a}_{is}$.

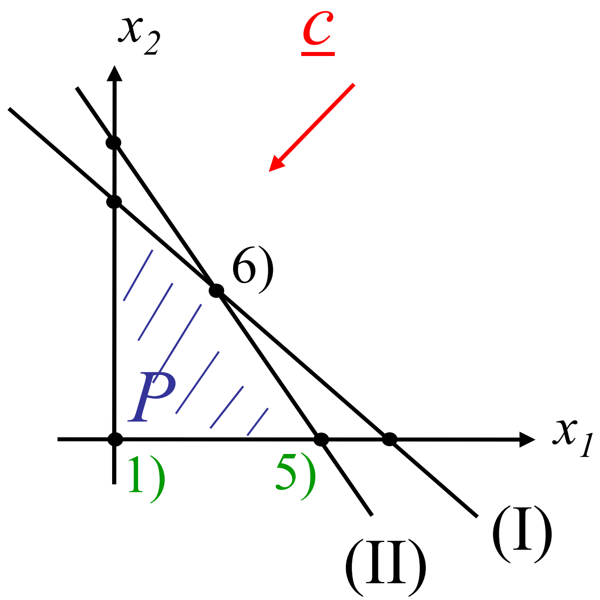
pivot

$$
\begin{array}{c}
 \\
r \rightarrow \\
 \\
 \\
\end{array}
\begin{array}{c}
\quad\; s \downarrow \quad A \\
\begin{bmatrix} 1 & 2 & 1 & -1 & 0 \\ 0 & 4 & -1 & 0 & 1 \\ 2 & 0 & 3 & 1 & 0 \end{bmatrix}
\end{array}
\begin{array}{c}
\underline{b} \\
\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}
\end{array}
\rightarrow
\begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ -2 & 0 & -3 & 2 & 1 \\ 2 & 0 & 3 & 1 & 0 \end{bmatrix}
\begin{bmatrix} 3/2 \\ -4 \\ 5 \end{bmatrix}
$$

do not affect the set of feasible solutions

Example:

$$\min \ -x_1 - x_2$$
$$\begin{aligned} \text{s.t.} \quad x_1 + x_2 + s_1 \quad\quad &= 6 \quad \text{(I)} \\ 2x_1 + x_2 \quad\quad + s_2 &= 8 \quad \text{(II)} \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$



Move from vertex 1) to vertex 5)

System in canonical form w.r.t. the basis with s_1 and s_2 basic (vertex 1):

$$\begin{aligned} s_1 &= 6 - x_1 - x_2 \\ s_2 &= 8 - 2x_1 - x_2 \end{aligned}$$

$$\begin{aligned} x_1 + x_2 + s_1 \quad\quad &= 6 \\ r \rightarrow 2x_1 + x_2 \quad\quad + s_2 &= 8 \end{aligned}$$

x_1 enters the basis and s_2 exits

pivot

$\uparrow$
s

columns of basic variables

System in canonical form w.r.t. the basis with x_1 and s_1 basic (vertex 5):

$$\begin{aligned} x_1 &= 4 - \tfrac{1}{2} x_2 - \tfrac{1}{2} s_2 \\ (4 - \tfrac{1}{2} x_2 - \tfrac{1}{2} s_2) + x_2 + s_1 &= 6 \end{aligned} \quad \Rightarrow \quad \begin{aligned} \tfrac{1}{2} x_2 + s_1 - \tfrac{1}{2} s_2 &= 2 \\ x_1 + \tfrac{1}{2} x_2 \quad\quad + \tfrac{1}{2} s_2 &= 4 \end{aligned}$$

Moving to an adjacent vertex (basic feasible solution)

Goals : i) improve the objective function value

ii) preserve feasibility

Choice of the pivot column s

1) Which non basic variable enters the basis?

- Any one with reduced cost $\bar{c}_j < 0$.
- One that yields the maximum Δz w.r.t. $z = \underline{c}_B^T B^{-1} \underline{b}$ (the actual decrement Δz also depends on θ^*).
- **Bland's rule** : $s = \min\{ j : \bar{c}_j < 0\}$.

For maximization problems : $\bar{c}_j > 0$

Choice of the pivot row r

otherwise no limit!

2) Which basic variable leaves the basis?

Min ratio test: index i with smallest $\dfrac{\bar{b}_i}{\bar{a}_{is}} = \theta^*$ among those with $\bar{a}_{is} > 0$.

Thightest upper bound on increase of x_s

- **Bland's rule** : $r = \min\{ i : \dfrac{\bar{b}_i}{\bar{a}_{is}} = \theta^*, \bar{a}_{is} > 0\}$
- randomly…

Unboundedness: If $\exists\ \bar{c}_j < 0$ with $\bar{a}_{ij} \leq 0 \ \ \forall i$, no element of the j-th column can play the role of a pivot.

$\Rightarrow$ The minimization problem is unbounded!

4.5.4 "Tableau" representation

System $\begin{cases} z = \underline{c}^T \underline{x} \\ A\underline{x} = \underline{b} \end{cases}$ with (implicit) nonnegativity constraints

Initial tableau:

- right hand side of the objective function

$$\begin{array}{r|c|c|l} & & x_1 \;\dots\; \qquad x_n & \\ \hline & 0 & \underline{c}^T & \leftarrow \text{objetive function} \\ \hline m \text{ rows} \left\{ \right. & \underline{b} & A & \end{array}$$

$\uparrow$

right hand side vector

Consider a basis B and a partition $A = [B\ N]$

	$x_1 \ \dots \ x_m$	$x_{m+1} \ \dots \ x_n$		z
0	$\underline{c}^T{}_B$	$\underline{c}^T{}_N$		-1
$\underline{b}$	B	N		$0 \ \vdots \ 0$

$0 = \underline{c}^T\underline{x} - z$

by "pivoting" operations (or pre-multiplying by B^{-1}) we put the <u>tableau</u> in <u>canonical form</u> with respect to B:

		$x_1 \ \dots \ x_m$	$x_{m+1} \ \dots \ x_n$
$-z$	$-z_0$	$0 \ \dots \ 0$	$\overline{\underline{c}}^T{}_N$
$x_{B[1]} \ \vdots \ x_{B[m]}$	$\overline{\underline{b}}$	I	$\overline{N}$

basic variables

$$z = \underbrace{\underline{c}^T{}_B B^{-1}\underline{b}}_{z_0} + \overline{\underline{c}}^T{}_N \underline{x}_N$$

$$\overline{\underline{b}} = B^{-1}\underline{b}$$

<u>Example</u>: min $z = -x_1 - x_2$

$$
\begin{aligned}
6x_1 + 4x_2 + x_3 &= 24 \\
3x_1 - 2x_2 + x_4 &= 6 \\
x_i \geq 0 \quad & i = 1, \dots, 4
\end{aligned}
$$

Tableau w.r.t. the basis with columns 3, 4:

$I_{2\times 2}$

$$
\begin{array}{rc|c|cccc|}
 & & & \overset{s}{\downarrow} & & & \\
 & & & x_1 & x_2 & x_3 & x_4 \\
\cline{3-7}
 & -z & 0 & -1 & -1 & 0 & 0 \\
\cline{3-7}
 & x_3 & 24 & 6 & 4 & 1 & 0 \\
r \longrightarrow & x_4 & 6 & 3 & -2 & 0 & 1 \\
\cline{3-7}
\end{array}
$$

pivot

basis

Pivot w.r.t. ③ amounts to deriving an expression for x_1 from the pivot row and substituting it in all other rows.

x_1 enters in the basis and x_4 exits the basis

Tableau w.r.t. the new basis:

		x_1	x_2	x_3	x_4
$-z$	2	0	-5/3	0	1/3
x_3	12	0	8	1	-2
x_1	2	1	-2/3	0	1/3

$\leftarrow$ reduced costs

basis

corresponding basic feasible solution:

$x^T = (2, 0, 12, 0)$ with $z = -2$.

$$\begin{pmatrix} x_1 \\ \cdot \\ 6 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ \cdot \\ 0 \\ 1 \end{pmatrix}$$

x_2 only non basic variable can "enter" the basis ($\bar{c}_2 = -5/3 < 0$)

x_3 only basic variable can "exit" the basis ($\bar{a}_{rs} = 8 > 0$)

		x_1	x_2 (s ↓)	x_3	x_4
$-z$	2	0	-5/3	0	1/3
$r \rightarrow x_3$	12	0	8	1	-2
x_1	2	1	-2/3	0	1/3

$\Rightarrow$

		x_1	x_2	x_3	x_4 (↑ s)
$-z$	9/2	0	0	5/24	-1/12
x_2	3/2	0	1	1/8	-1/4
x_1 $\leftarrow r$	3	1	0	1/12	1/6

$\Rightarrow$

		x_1	x_2	x_3	x_4
$-z$	6	½	0	¼	0
x_2	6	3/2	1	¼	0
x_4	18	6	0	½	1

All reduced costs ≥ 0

$\Rightarrow$ **optimal basic (feasible) solution**:

$\underline{x}^{*T} = (0, 6, 0, 18)$ with $z^* = -6$

Simplex algorithm (LP with min)

```
BEGIN
  Let B[1],…,B[m] be the column indices of the inital feasible basis B;
  Construct the initial tableau Ā = {ā[i,j]: 0≤i≤m, 0≤j≤n} in canonical
                                                         form w.r.t. B;
  unbounded := false;  optimal := false;
  WHILE (optimal = false) AND (unbounded = false) THEN
    IF ā[0,j] ≥ 0 ∀ j=1,…,m THEN optimal := true; /* for LP with min */
    ELSE                                               reduced costs
      Select a non basic variable x_s with ā[0,s] < 0;
      IF ā[i,s] ≤ 0 ∀ i=1,…,m  THEN  unbounded := true;
        ELSE
          Determine index r that minimizes ā[i,0] / ā[i,s]
          with 1 ≤ i ≤ m and ā[i,s] > 0;
          pivot(r,s)     /* update tableau */
          B[r] := s;
        END-IF
    END-IF
  END
```

Procedure pivot(r,s)?

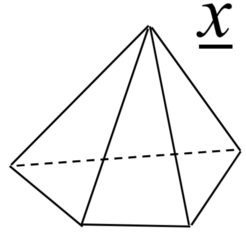
4.5.5 Degenerate basic feasible solutions and convergence

Definition: A *basic feasible solution* $\underline{x}$ is *degenerate* if it contains at least one basic variable =0.

$\underline{x}$ with more than n-m zeroes correspond to several distinct bases!

Same vertex:

$\underline{x}$



More than n constraints (the m of $A\underline{x} = \underline{b}$ and more than n-m among the n of $\underline{x} \geq \underline{0}$) are satisfied with equality ("active").

In the presence of degenerate basic feasible solutions (BFSs), a basis change may not decrease the objective function value:

If the current BFS is degenerate, one can have $\theta^*=0$ and hence the new BFS is identical (same vertex).

Note that a degenerate BFS can arise from a non degenerate one: even if $\theta^* > 0$, several basic variables may go to 0 when x_s is increased to θ^*.

$\Rightarrow$ One can cycle through a sequence of "degenerate" bases associated to the same vertex.

Several “anticycling” rules have been proposed for the choice of the variables that enter and exit the bases (indices r and s).

Bland’s rule: Among all candidate variables to enter/exit the basis (x_s / x_r) always select the one with smallest index.

Robert. Bland

Proposition: The Simplex algorithm with Bland’s rule terminates after $\leq \binom{n}{m}$ iterations.

finite number of pivots

In some "pathological" cases (see e.g. Klee & Minty 72), the number of iterations grows exponentially w.r.t. n and/or m.

However the **Simplex algorithm** is overall **very efficient**.

Extensive experimental campaigns:

The number of iterations grows linearly w.r.t. m ($m \leq . \leq 3m$) and very slowly ($\approx$ logarithmically) w.r.t. n.

4.5.6 Two-phase simplex method

Phase 1: Determine an intial basic feasible solution.

Example:

$$
\begin{array}{llll}
\min & z = x_1 + x_3 & & \\
& x_1 + 2x_2 \leq 5 & \rightarrow & x_1 + 2x_2 + x_4 = 5 \\
& x_2 + 2x_3 = 6 & & \\
& x_1, x_2, x_3 \geq 0 & & x_4 \geq 0
\end{array}
$$

$\nexists$ a submatrix I_{2x2} of A!

Given (P)

$$
\begin{array}{ll}
\min & z = \underline{c}^T \underline{x} \\
& A\underline{x} = \underline{b} \\
& \underline{x} \geq \underline{0}
\end{array}
$$

Assumption: $\underline{b} \geq \underline{0}$

Auxilliary problem with artificial variables y_i, $1 \leq i \leq m$,

$$(P_A) \quad \begin{array}{ll} \min & v = \sum_{i=1}^{m} y_i \\ & A\,\underline{x} + I\,\underline{y} = \underline{b} \\ & \underline{x} \geq \underline{0},\ \underline{y} \geq \underline{0} \end{array}$$

$\exists$ an obvious initial basic feasible solution $\underline{y} = \underline{b} \geq \underline{0}$ and $\underline{x} = \underline{0}$

1) If $v^* > 0$, then (P) is infeasible.
2) If $v^* = 0$, clearly $\underline{y}^* = \underline{0}$ and $\underline{x}^*$ is a basic feasible solution of (P).

For 2) there are two cases:

- If y_i is non basic $\forall i$, with $1 \leq i \leq m$, delete the corresponding columns and obtain a tableau in canonical form w.r.t. a basis; the row of z must be determined by substitution.

- If $\exists$ a basic y_i (the basic feasible solution is degenerate), we perform a «pivot» operation w.r.t. a coefficient $\neq 0$ of the row of y_i so as to "exchange" y_i with a non basic variable x_j.

cf. example

Example:

$$(P)\quad \begin{aligned} \min \quad & z = x_1 + x_2 + 10\, x_3 \\ & x_2 + 4x_3 = 2 \\ & -2x_1 + x_2 - 6x_3 = 2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

$$(P_A)\quad \begin{aligned} \min \quad & v = y_1 + y_2 \\ & x_2 + 4x_3 + y_1 = 2 \\ & -2x_1 + x_2 - 6x_3 + y_2 = 2 \\ & x_1, x_2, x_3, y_1, y_2 \geq 0 \end{aligned}$$

Put $v = y_1 + y_2$ in canonical form by substituting the expression of y_1 and y_2 in terms of x_1, x_2 and x_3.

		x_1	x_2	x_3	y_1	y_2
$-v$	-4	2	-2	2	0	0
y_1	2	0	1	4	1	0
y_2	2	-2	1	-6	0	1

		x_1	x_2	x_3	y_1	y_2
$-v$	-4	2	-2	2	0	0
y_1	2	0	1	4	1	0
y_2	2	-2	1	-6	0	1

↓

		x_1	x_2	x_3	y_1	y_2
$-v$	0	2	0	10	2	0
x_2	2	0	1	4	1	0
y_2	0	-2	0	-10	-1	1

optimal value $v^* = 0$

$\underline{x}^* = (0, 2, 0)$

$\underline{y}^* = (0, 0)$

By selecting as «pivot» the coefficient –2 of the row of y_2, we obtain:

Equivalent optimal basis

		x_1	x_2	x_3	y_1	y_2
$-v$	0	0	0	0	1	1
x_2	2	0	1	4	1	0
x_1	0	1	0	5	½	-½

The column $\begin{matrix} 0 \\ 1 \end{matrix}$ of I has been transferred in the "area" the original x_j variables.

$\Rightarrow$ optimal basic feasible solution of (P_A)

$\underline{x} = (0, 2, 0)$

is a basic feasible solution of (P).

$$z = x_1 + x_2 + 10\, x_3 \qquad \neq \text{ canonical form}$$

non basic variable

By substituting:

$$\begin{cases} x_2 = 2 - 4x_3 \\ x_1 = \quad - 5x_3 \end{cases} \quad \Rightarrow \quad z = 2 + x_3$$

		x_1	x_2	x_3
$-z$	-2	0	0	1
x_2	2	0	1	4
x_1	0	1	0	5

Tableau corresponding to the initial basic feasible solution of (P).

Since the basic feasible solution found is (already) optimal, here no need for the second phase!

4.5.7 Polynomial-time algorithms for LP

- Ellipsoid method (L. Khachiyan 1979)

  Theoretically important.

- Interior point methods (N. Karmarkar 1984,…)

Narendra Karmarkar (1957-)

Very efficient variants (e.g. barrier methods) for some types of instances (e.g. sparse and large-scale).