4.7 Sensitivity analysis in Linear Programming

Evaluate the "sensitivity" of an optimal solution with respect to variations in the data (model parameters).

Example: Production planning

profit

availability *i*-th resource

$$\max \quad \sum_{j=1}^{n} p_j x_j$$

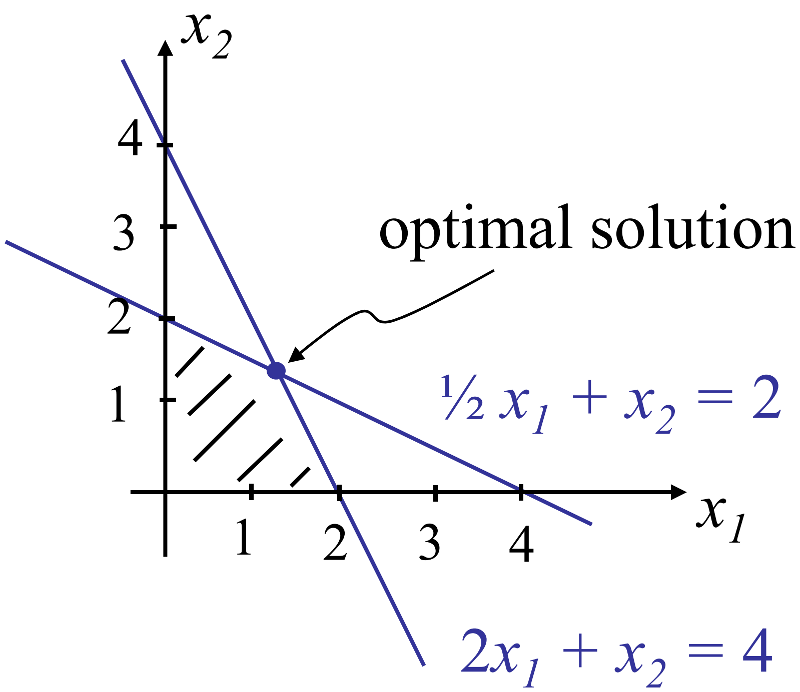
$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad 1 \leq i \leq m$$

$$x_j \geq 0 \qquad 1 \leq j \leq n$$

4.7.1 Geometric interpretation

Example: max $x_1 + x_2$

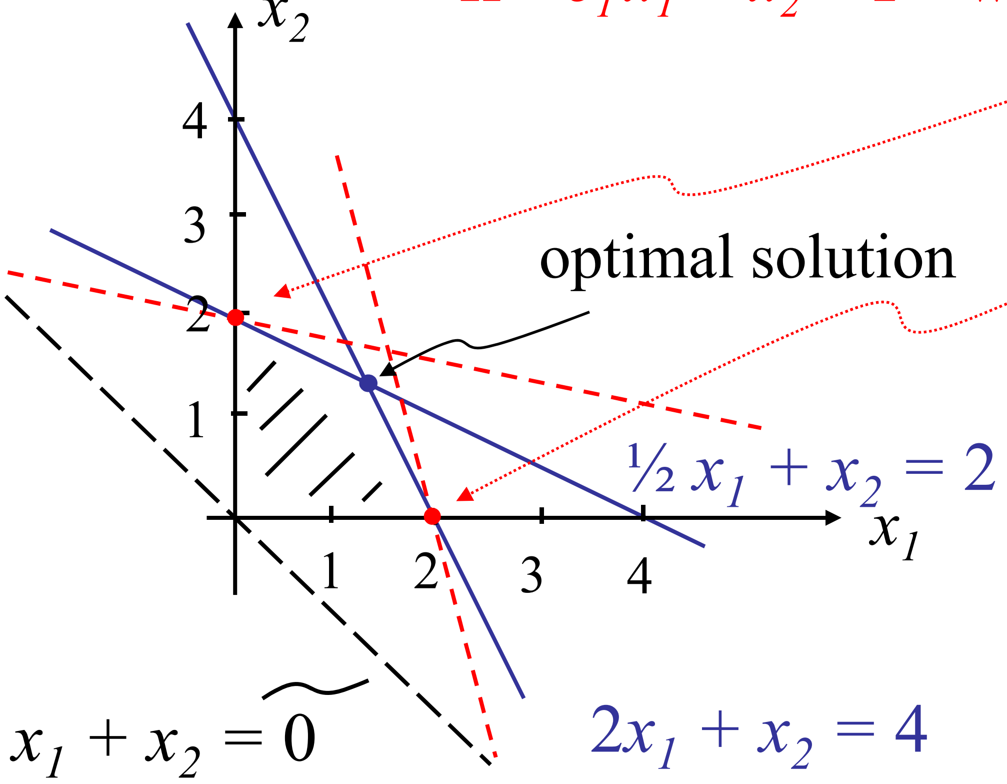
$$\tfrac{1}{2} x_1 + x_2 \le 2$$
$$2 x_1 + x_2 \le 4$$
$$x_1, x_2 \ge 0$$



$$\underline{x}^* = \begin{bmatrix} 4/3 \\ 4/3 \end{bmatrix} \text{ with } z^* = 8/3$$

Variation of the objective function coefficients

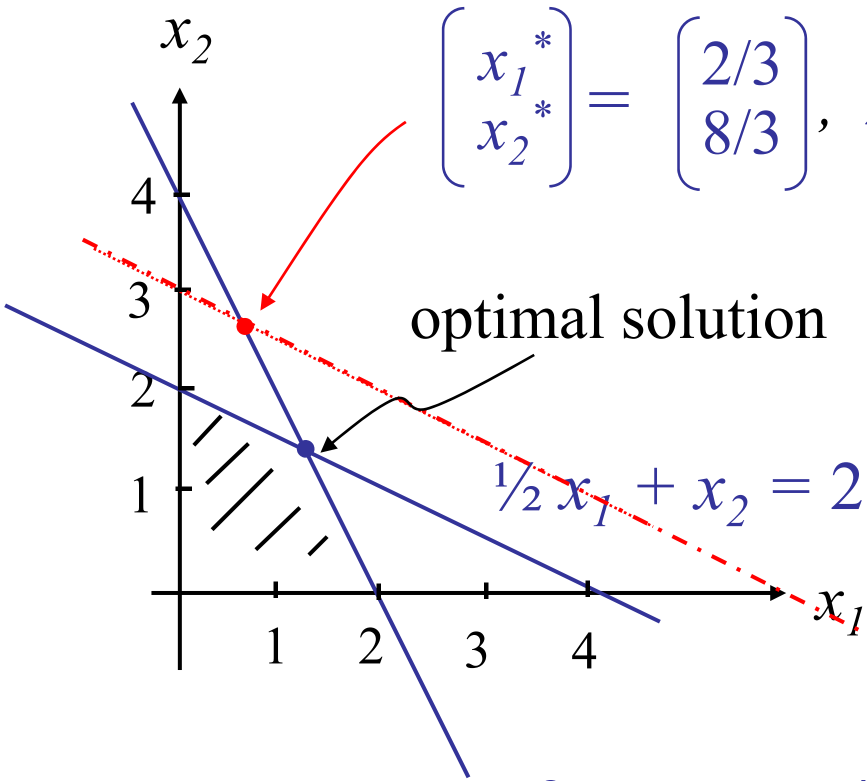
If $c_1 x_1 + x_2 = z$ with $½ \leq c_1 \leq 2$



optimal solution $\underline{x}^* = \begin{pmatrix} 4/3 \\ 4/3 \end{pmatrix}$ with $z^* = 8/3$

$$\begin{aligned} \max \quad & x_1 + x_2 \\ & ½\, x_1 + x_2 \leq 2 \\ & 2\, x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Variations of the right-hand-side terms



$$\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 2/3 \\ 8/3 \end{pmatrix}, \quad z^* = 2/3 + 8/3 = 10/3$$

optimal solution $\begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \begin{pmatrix} 4/3 \\ 4/3 \end{pmatrix}$ with $z^* = 8/3$

$$\tfrac{1}{2} x_1 + x_2 = 2 + 1$$

$$2x_1 + x_2 = 4$$

$$\begin{array}{ll} \max & x_1 + x_2 \\ & \tfrac{1}{2} x_1 + x_2 \leq 2+1 \\ & 2 x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{array}$$

<u>Definition</u>:

The <u>*shadow price*</u> of the *i*-th resourse = maximum <u>price</u> the company is willing to pay to buy an <u>additional unit</u> of <u>*i*-th resource</u>.

$$\text{Shadow price of the first resource} = \underbrace{10/3}_{} - 8/3 = 2/3$$

$$8/3 = z^*$$

4.7.2 Sensitivity analysis: algebraic form

Evaluate the "sensitivity" of an optimal solution when the model parameters (c_j, a_{ij} , b_i) vary.

Given $\min \ \underline{c}^T \underline{x}$ and optimal basic solution $\underline{x}^*$

$$(P) \quad \begin{aligned} A\underline{x} &= \underline{b} \\ \underline{x} &\geq \underline{0} \end{aligned} \qquad\qquad \begin{aligned} \underline{x}_B^* &= B^{-1}\underline{b} \geq \underline{0} \\ \underline{x}_N^* &= \underline{0} \end{aligned}$$

Within which limits the basis B remains optimal?

Conditions: 1) $B^{-1}\underline{b} \geq \underline{0}$ feasibility

2) $\underline{\bar{c}}^T_N = \underline{c}^T_{\mathrm{N}} - \underline{c}^T_B B^{-1} N \geq \underline{0}^T$ optimality

Topic of the **4th comupter laboratory** session.

Variation of right-hand-side terms

$$\underline{b}' := \underline{b} + \delta_k \, \underline{e}_k \quad \text{with} \quad \underline{e}_k = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow k \qquad 1 \leq k \leq \text{n}$$

The basis B with the basic solution

$$\underline{x}^* = \begin{pmatrix} B^{-1}(\underline{b} + \delta_k \, \underline{e}_k) \\ \underline{0} \end{pmatrix}$$

$\underline{\bar{c}}_N$ do not vary

remains optimal as long as

$$B^{-1}(\underline{b} + \delta_k \, \underline{e}_k) \geq \underline{0} \quad \Rightarrow \quad \underbrace{B^{-1}\underline{b} \geq -\delta_k \, B^{-1}\underline{e}_k}$$

m inequalities that define an interval of variation for δ_k

The basis B remains optimal, but the optimal basic feasible solution changes.

The objective function values goes from $\underline{c}^T_B B^{-1}\underline{b}$ to $\underline{c}^T_B B^{-1}(\underline{b} + \delta_k \underline{e}_k)$

$$\Rightarrow \quad \Delta z^* = \underbrace{\underline{c}^T_B B^{-1}}_{\underline{y}^{*T}}(\delta_k \underline{e}_k) = \delta_k \underbrace{y^*_k}_{\text{shadow price!}} \leftarrow \text{optimal value of the dual variable}$$

$$y^*_k = \frac{\partial z^*}{\partial b_k}$$

Variation of the cost coefficients

Given $\underline{c}' := \underline{c} + \delta_k \underline{e}_k$

a basis B remains optimal as long as

$$\bar{\underline{c}}'^T_N := \underline{c}'^T_N - \underline{c}'^T_B B^{-1} N \geq \underline{0}$$

In such a case, the optimal basic (feasible) solution does not change:

$$\begin{cases} \underline{x}^*_B = B^{-1}\underline{b} \\ \underline{x}^*_N = \underline{0} \end{cases}$$

If x_k is a <u>nonbasic variable</u>

$$\underline{\bar{c}}'^T_N = \underline{c}'^T_N - \underline{c}'^T_B B^{-1} N \geq \underline{0}^T \qquad \text{con} \quad \underline{c}'^T_B = \underline{c}^T_B$$

$$\underline{\bar{c}}'^T_N = (\underline{c}^T_N + \delta_k \underline{e}^T_k) - \underline{c}^T_B B^{-1} N$$

$$= (\underline{c}^T_N - \underline{c}^T_B B^{-1} N) + \delta_k \underline{e}^T_k$$

$$= \underline{\bar{c}}^T_N + \delta_k \underline{e}^T_k \geq \underline{0}^T$$

$$\Rightarrow \quad \delta_k \geq - \bar{c}_k$$

Reduced cost = max decrease of c_k for which B remains optimal (larger decrease $\Rightarrow \bar{c}'_k < 0$)

$\Delta z^* = 0$ since $z^* = \underline{c}^T_B B^{-1} \underline{b} = \underline{c}^T_B \underline{x}^*_B$

If x_k is a basic variable

$$\underline{\bar{c}}'^T_N = \underline{c}'^T_N - \underline{c}'^T_B B^{-1} N \geq \underline{0}^T \qquad \text{with} \quad \underline{c}'^T_N = \underline{c}^T_N$$

$$\underline{\bar{c}}'^T_N = \underline{c}^T_N - (\underline{c}^T_B + \delta_k \underline{e}^T_k) B^{-1} N$$

$$= (\underline{c}^T_N - \underline{c}^T_B B^{-1} N) - \delta_k \underline{e}^T_k B^{-1} N$$

$$\Rightarrow \underline{\bar{c}}^T_N - \delta_k \underline{\rho}^T_k \geq \underline{0}^T$$

where $\underline{\rho}^T_k$ k-the row of $B^{-1}N$

$$\underbrace{\underline{\bar{c}}^T_N \geq \delta_k \underline{\rho}^T_k}$$

n-m inequalities that define a variation interval for δ_k

In such a case

$$\Delta z^* = \delta_k \underline{e}^T{}_k \underbrace{B^{-1}\underline{b}}_{\underline{x}^*{}_B} = \delta_k x^*{}_k \longleftarrow \frac{\partial z^*}{\partial c_k}$$

Similar analysis for the coefficients a_{ij}

Example: sensitivity analysis

Initial tableau:

$$\min \quad \underline{c}^T \underline{x}$$
$$A\underline{x} = \underline{b}$$
$$\underline{x} \geq \underline{0}$$

initial basis: x_4, x_5, x_6

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	0	-3	-1	-3	0	0	0
x_4	2	2	1	1	1	0	0
x_5	5	1	2	3	0	1	0
x_6	6	2	2	1	0	0	1

↑ $\underline{b}$ (second column); I (columns x_4, x_5, x_6)

Optimal tableau:

$$\underline{\bar{c}}^T_N := \underline{c}^T_N - \underline{c}^T_B B^{-1} N$$

"pivoting" operations ≡ premultiply by B^{-1}

		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	27/5	0	7/5	0	6/5	3/5	0
x_1	1/5	1	1/5	0	3/5	-1/5	0
x_3	8/5	0	3/5	1	-1/5	2/5	0
x_6	4	0	1	0	-1	0	1

$B^{-1}\underline{b}$ (column 0)

$B^{-1}N$ (columns x_2, x_4, x_5)

$B^{-1}I = B^{-1}$ (columns x_5, x_6 ...)

Thus

$$\underline{x}_B = \begin{pmatrix} x_1 \\ x_3 \\ x_6 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 2 & 1 & 1 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 3/5 & -1/5 & 0 \\ -1/5 & 2/5 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$B^{-1}\underline{b} = \begin{pmatrix} 1/5 \\ 8/5 \\ 4 \end{pmatrix} \qquad \underline{c}_B = \begin{pmatrix} -3 \\ -3 \\ 0 \end{pmatrix} \qquad \underline{x}_N = \begin{pmatrix} x_2 \\ x_4 \\ x_5 \end{pmatrix} \quad \underline{c}_N = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

Optimal dual solution:

$$\underline{y}^T = \underline{c}^T{}_B B^{-1} = \begin{pmatrix} -3 & -3 & 0 \end{pmatrix} \begin{pmatrix} 3/5 & -1/5 & 0 \\ -1/5 & 2/5 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -6/5 & -3/5 & 0 \end{pmatrix}$$

- <u>Variations of $\underline{b}$</u> : $\underline{b}' := \underline{b} + \delta_k \underline{e}_k$

$$\underline{e}_k = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \leftarrow k$$

$$\underline{x}^*_B = \begin{pmatrix} B^{-1}(\underline{b} + \delta_k \underline{e}_k) \\ \underline{0} \end{pmatrix}$$ remains optimal

as long as $B^{-1}(\underline{b} + \delta_k \underline{e}_k) \geq \underline{0} \;\Rightarrow\; B^{-1}\underline{b} \geq - \delta_k B^{-1} \underline{e}_k$

$k = 1, 2, 3$

$$\underline{k=1}: \begin{pmatrix} 1/5 \\ 8/5 \\ 4 \end{pmatrix} \geq - \delta_1 \begin{pmatrix} 3/5 \\ -1/5 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} -1/3 \leq \delta_1 \\ 8 \geq \delta_1 \\ 4 \geq \delta_1 \end{cases} \Rightarrow -1/3 \leq \delta_1 \leq 4$$

$\underline{k=2}$: $-4 \leq \delta_2 \leq 1$ $\qquad$ $\underline{k=3}$: $-4 \leq \delta_3$

- Variations of $\underline{c}$: $\underline{c}' := \underline{c} + \delta_k \underline{c}_k$

For x_k nonbasic, B remains otpimal if and only if $\delta_k \geq -\bar{c}_k$

$$\delta_2 \geq -7/5$$
$$\delta_4 \geq -6/5$$
$$\delta_5 \geq -3/5$$

reduced cost

For x_k basic, B remains optimal if and only if

$$\underline{\bar{c}}^T_N \geq \delta_k \underline{e}^T_k B^{-1} N$$

$$\underline{\bar{c}}_N = \begin{pmatrix} 7/5 \\ 6/5 \\ 3/5 \end{pmatrix} \qquad B^{-1}N = \begin{pmatrix} 3/5 & -1/5 & 0 \\ -1/5 & 2/5 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1/5 & 3/5 & -1/5 \\ 3/5 & -1/5 & 2/5 \\ 1 & -1 & 0 \end{pmatrix}$$

$\underline{k = 1}$:

$$\begin{pmatrix} 7/5 & 6/5 & 3/5 \end{pmatrix} \geq \delta_1 \begin{pmatrix} 1/5 & 3/5 & -1/5 \end{pmatrix}$$

$$-3 \leq \delta_1 \leq 2$$

$\underline{k = 3}$:

$$\begin{pmatrix} 7/5 & 6/5 & 3/5 \end{pmatrix} \geq \delta_3 \begin{pmatrix} 3/5 & -1/5 & 2/5 \end{pmatrix}$$

$$-6 \leq \delta_3 \leq 3/2$$

$\underline{k = 6}$:

$$\begin{pmatrix} 7/5 & 6/5 & 3/5 \end{pmatrix} \geq \delta_6 \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}$$

$$-6/5 \leq \delta_6 \leq 7/5$$