4.6 Linear Programming duality

To any minimization (maximization) LP we can associate a closely related maximization (minimization) LP based on same parameters.

Different spaces and objective functions but in general the optimal objective function values coincide.

Example: The value of a maximum feasible flow is equal to the capacity of a cut (separating the source *s* and the sink *t*) of minimum capacity.

Motivation: estimate of the optimal value

Given

$$
\begin{array}{lrl}
\max & z = 4x_1 + x_2 + 5x_3 + 3x_4 & \\
& x_1 - x_2 - x_3 + 3x_4 \leq 1 & \text{(I)} \\
& 5x_1 + x_2 + 3x_3 + 8x_4 \leq 55 & \text{(II)} \\
& -x_1 + 2x_2 + 3x_3 - 5x_4 \leq 3 & \text{(III)} \\
& x_i \geq 0 & i = 1, \ldots, 4
\end{array}
$$

find an estimate of the optimal value z^*.

Any feasible solution is a lower bound.

Lower bounds: (0,0,1,0) → $z^* \geq 5$

(2,1,1,1/3) → $z^* \geq 15$

(3,0,2,0) → $z^* \geq 22$

… …

Even if we are lucky, we are not sure it is the optimal solution!

Upper bounds:

- By multiplying $5x_1 + x_2 + 3x_3 + 8x_4 \leq 55$ (II) by 5/3, we obtain an inequality that dominates the objective function:

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq 25/3x_1 + 5/3x_2 + 5x_3 + 40/3x_4 \leq 275/3$$

$\forall$ feasible solution

$$\Rightarrow \quad z^* \leq 275/3.$$

- By adding constraints (II) and (III), we obtain:

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq 4x_1 + 3x_2 + 6x_3 + 3x_4 \leq 58$$

$\Rightarrow \quad z^* \leq 58$ better upper bound.

Linear combinations with nonnegative multipliers of inequality constraints yields valid inequalities

General strategy: Linearly combine the constraints with non negative multiplicative factors (*i*-th constraint multiplied by $y_i \geq 0$).

first case: $y_1=0, y_2=5/3, y_3=0$

second case: $y_1=0, y_2=1, y_3=1$

In general any such linear combination of (I), (II), (III) reads

$$y_1(x_1 - x_2 - x_3 + 3x_4) + y_2(5x_1 + x_2 + 3x_3 + 8x_4)$$
$$+ y_3(-x_1 + 2x_2 + 3x_3 - 5x_4) \leq y_1 + 55y_2 + 3y_3$$

which is equivalent to:

$$(y_1 + 5y_2 - y_3)\, x_1 + (-y_1 + y_2 + 2y_3)\, x_2 + (-y_1 + 3y_2 + 3y_3)\, x_3 + (3y_1 + 8y_2 - 5y_3)\, x_4 \leq y_1 + 55y_2 + 3y_3 \qquad (*)$$

Observation: $y_i \geq 0$ so that the inequality direction is unchanged.

To use the left hand side of (*) as upper bound on

$$z = 4x_1 + x_2 + 5x_3 + 3x_4$$

$$z = 4x_1 + x_2 + 5x_3 + 3x_4$$

we must have

$$\begin{cases} y_1 + 5y_2 - y_3 \geq 4 \\ -y_1 + y_2 + 2y_3 \geq 1 \\ -y_1 + 3y_2 + 3y_3 \geq 5 \\ 3y_1 + 8y_2 - 5y_3 \geq 3 \end{cases} \qquad y_i \geq 0, \quad i = 1, 2, 3.$$

In such a case, any feasible solution $\underline{x}$ satisfies

$$4x_1 + x_2 + 5x_3 + 3x_4 \leq y_1 + 55y_2 + 3y_3$$

In particular: $z^* \leq y_1 + 55y_2 + 3y_3$

Since we look for the best possible upper bound on z^*:

Original problem:

$$\max \; z = 4x_1 + x_2 + 5x_3 + 3x_4$$

$$\begin{array}{rcl} x_1 - x_2 - x_3 + 3x_4 & \leq 1 & \text{(I)} \\ 5x_1 + x_2 + 3x_3 + 8x_4 & \leq 55 & \text{(II)} \\ -x_1 + 2x_2 + 3x_3 - 5x_4 & \leq 3 & \text{(III)} \\ x_i \geq 0 & i = 1, \ldots, 4 & \end{array}$$

$$(D) \quad \begin{array}{ll} \min & y_1 + 55y_2 + 3y_3 \\ & y_1 + 5y_2 - y_3 \geq 4 \\ & -y_1 + y_2 + 2y_3 \geq 1 \\ & -y_1 + 3y_2 + 3y_3 \geq 5 \\ & 3y_1 + 8y_2 - 5y_3 \geq 3 \\ & y_i \geq 0 \qquad i = 1, 2, 3 \end{array}$$

Definition: The problem (*D*) is the *dual problem*, while the original problem is the *primal problem*.

In matrix form:

Primal (P)

$$\begin{aligned} \max \quad & z = \underline{c}^T \underline{x} \\ & A\underline{x} \leq \underline{b} \\ & \underline{x} \geq \underline{0} \end{aligned}$$

Dual (D)

$$\begin{aligned} \min \quad & w = \underline{b}^T \underline{y} \\ & A^T\underline{y} \geq \underline{c} \qquad \text{or} \quad \underline{y}^T A \geq \underline{c}^T \\ & \underline{y} \geq \underline{0} \end{aligned}$$

Dual problem

$$(P) \quad \begin{array}{ll} \max & z = \underline{c}^T \underline{x} \\ & A\underline{x} \leq \underline{b} \\ & \underline{x} \geq \underline{0} \end{array} \qquad\qquad (D) \quad \begin{array}{ll} \min & w = \underline{b}^T \underline{y} \\ & A^T\underline{y} \geq \underline{c} \\ & \underline{y} \geq \underline{0} \end{array}$$

Dual of an LP in standard form ?

$$\begin{array}{ll} \min & z = \underline{c}^T \underline{x} \\ & A\underline{x} = \underline{b} \\ & \underline{x} \geq \underline{0} \end{array}$$

Standard form:

$$(P)\quad \begin{array}{ll} \min & z = \underline{c}^T\underline{x} \\ & A\underline{x} = \underline{b} \\ & \underline{x} \geq \underline{0} \end{array} \qquad \equiv \qquad \begin{array}{ll} -\max & -\underline{c}^T\underline{x} \\ & \left.\begin{pmatrix} A \\ -A \end{pmatrix} \underline{x} \leq \begin{pmatrix} \underline{b} \\ -\underline{b} \end{pmatrix}\right\} A'\underline{x} \leq \underline{b}' \\ & \underline{x} \geq \underline{0} \end{array}$$

with A an $m\times n$ matrix

$$\Big\downarrow \ \text{dual}$$

$$\begin{array}{ll} -\min & (\underline{b}^T\ \ -\underline{b}^T)\begin{pmatrix} \underline{y}^1 \\ \underline{y}^2 \end{pmatrix} \qquad \Big\}\ \underline{y}' = \begin{pmatrix} \underline{y}^1 \\ \underline{y}^2 \end{pmatrix} \\ & \underbrace{(A^T\ \ -A^T)}_{A'^T}\begin{pmatrix} \underline{y}^1 \\ \underline{y}^2 \end{pmatrix} \geq -\underline{c} \\ & \underline{y}^1 \geq \underline{0}\,,\ \underline{y}^2 \geq \underline{0} \end{array}$$

$$
\begin{aligned}
-\min \quad & (\underline{b}^T \;\; -\underline{b}^T)\begin{pmatrix}\underline{y}^1\\ \underline{y}^2\end{pmatrix}\\
& (A^T \;\; -A^T)\begin{pmatrix}\underline{y}^1\\ \underline{y}^2\end{pmatrix} \geq -\underline{c}\\
& \underline{y}^1 \geq \underline{0}\,,\ \underline{y}^2 \geq \underline{0}
\end{aligned}
$$

$$\rotatebox{90}{$\equiv$}$$

$$
(D) \qquad
\begin{aligned}
\max \quad & w = \underline{b}^T\underline{y}\\
& A^T\underline{y} \leq \underline{c}\\
& \underline{y} \in \mathrm{R}^m
\end{aligned}
\qquad \equiv \qquad
\begin{aligned}
-\min \quad & -\underline{b}^T(\underline{y}^2 - \underline{y}^1)\\
& -A^T(\underline{y}^2 - \underline{y}^1) \geq -\underline{c}\\
& \underline{y}^1 \geq \underline{0}\,,\ \underline{y}^2 \geq \underline{0}
\end{aligned}
$$

$$\underline{y} := \underline{y}^2 - \underline{y}^1$$

unrestricted in sign!

Property: The dual of the dual problem coincides with the primal problem.

$$
(P)\quad \begin{array}{ll} \max & z = \underline{c}^T \underline{x} \\ & A\underline{x} \leq \underline{b} \\ & \underline{x} \geq \underline{0} \end{array}
\qquad\qquad
(D)\quad \begin{array}{ll} \min & w = \underline{b}^T \underline{y} \\ & A^T\underline{y} \geq \underline{c} \\ & \underline{y} \geq \underline{0} \end{array}
\quad \cdots
$$

Observation: it doesn't matter which one is a maximum or minimum problem.

General transformation rules

Primal (minimization)	Dual (maximization)
m constraints	m variables
n variables	n constraints
coefficients obj. fct	right hand side
right hand side	coefficients obj. fct
A	A^T
equality constraints	unrestriced variables
unrestriced variables	equality constraints
inequality constraints $\geq$ ($\leq$)	variables ≥ 0 (≤ 0)
variables ≥ 0 (≤ 0)	inequality constraints $\leq$ ($\geq$)

Example:

$$(P) \quad \begin{array}{rr} \max & x_1 + x_2 \\ & x_1 - x_2 \leq 2 \\ & 3x_1 + 2x_2 \geq 12 \\ & x_1, x_2 \geq 0 \end{array}$$

$$\begin{array}{rr} \max & x_1 + x_2 \\ & x_1 - x_2 \leq 2 \\ & -3x_1 - 2x_2 \leq -12 \\ & x_1, x_2 \geq 0 \end{array} \quad \xrightarrow[\text{dual}]{} \quad \begin{array}{rr} \min & 2y_1 - 12y_2 \\ & y_1 - 3y_2 \geq 1 \\ & -y_1 - 2y_2 \geq 1 \\ & y_1, y_2 \geq 0 \end{array}$$

Example: using the above rules

$$(P)\quad \begin{array}{rr} \max & x_1 + x_2 \\ & x_1 - \ \ x_2 \le 2 \\ & 3\,x_1 + 2x_2 \ge 12 \\ & x_1, x_2 \ge 0 \end{array} \quad \xrightarrow[\text{dual}]{} \quad \begin{array}{rl} \min & 2y_1 + 12y_2 \\ & y_1 + 3y_2 \ge 1 \\ & -y_1 + 2y_2 \ge 1 \\ & y_1 \ge 0,\ y_2 \le 0 \end{array}$$

$$\tilde{y}_2 := -y_2$$

$$\begin{array}{rl} \min & 2y_1 - 12\tilde{y}_2 \\ & y_1 - 3\tilde{y}_2 \ge 1 \\ & -\,y_1 - 2\tilde{y}_2 \ge 1 \\ & y_1 \ge 0,\ \tilde{y}_2 \ge 0 \end{array}$$

Exercise:

$$
(P)\quad \begin{array}{llr}
\min & 10x_1 + 20x_2 + 30x_3 & \\
 & 2x_1 - x_2 & \geq 1 \\
 & x_2 + x_3 & \leq 2 \\
 & x_1 - x_3 & = 3 \\
 & x_1 \geq 0,\ x_2 \leq 0,\ x_3 \text{ unrestricted} &
\end{array}
$$

Dual?

Weak duality theorem

$$\begin{array}{llcll} \min & z = \underline{c}^T\underline{x} & \qquad & \max & w = \underline{b}^T\underline{y} \\ (P) & A\underline{x} \geq \underline{b} & & (D) & A^T\underline{y} \leq \underline{c} \\ & \underline{x} \geq \underline{0} & & & \underline{y} \geq \underline{0} \end{array}$$

$$X := \{\underline{x} : A\underline{x} \geq \underline{b}\,,\ \underline{x} \geq \underline{0}\} \neq \emptyset \quad \text{and}$$

$$Y := \{\underline{y} : A^T\underline{y} \leq \underline{c}\,,\ \underline{y} \geq \underline{0}\} \neq \emptyset,$$

For each feasible solution $\underline{\bar{x}} \in X$ of (P) and each feasible solution $\underline{\bar{y}} \in Y$ of (D) we have

$$\underline{b}^T\overline{\underline{y}} \leq \underline{c}^T\overline{\underline{x}}.$$

Proof

For every pair $\underline{\bar{x}} \in X$ and $\underline{\bar{y}} \in Y$, we have $A\underline{\bar{x}} \ge \underline{b}$, $\underline{\bar{x}} \ge \underline{0}$ and

$A^T\underline{\bar{y}} \le \underline{c}$, $\underline{\bar{y}} \ge \underline{0}$ which imply that

$$\underset{\substack{\rotatebox[origin=c]{-90}{$\le$} \\ \underline{\bar{x}}^T A^T}}{\underline{b}^T}\underline{\bar{y}} \;\le\; \underline{\bar{x}}^T \underbrace{A^T\underline{\bar{y}}}_{\substack{\rotatebox[origin=c]{-90}{$\le$} \\ \underline{c}}} \le \; \underline{\bar{x}}^T\underline{c} = \underline{c}^T\underline{\bar{x}}$$

Consequence:

If $\underline{\bar{x}}$ is a feasible solution of (P) $(\underline{\bar{x}} \in X)$, $\underline{\bar{y}}$ is a feasible solution of (D) $(\underline{\bar{y}} \in Y)$,

and the values of the respective objective functions coincide

$$\underline{c}^T \underline{\bar{x}} = \underline{b}^T \underline{\bar{y}},$$

then

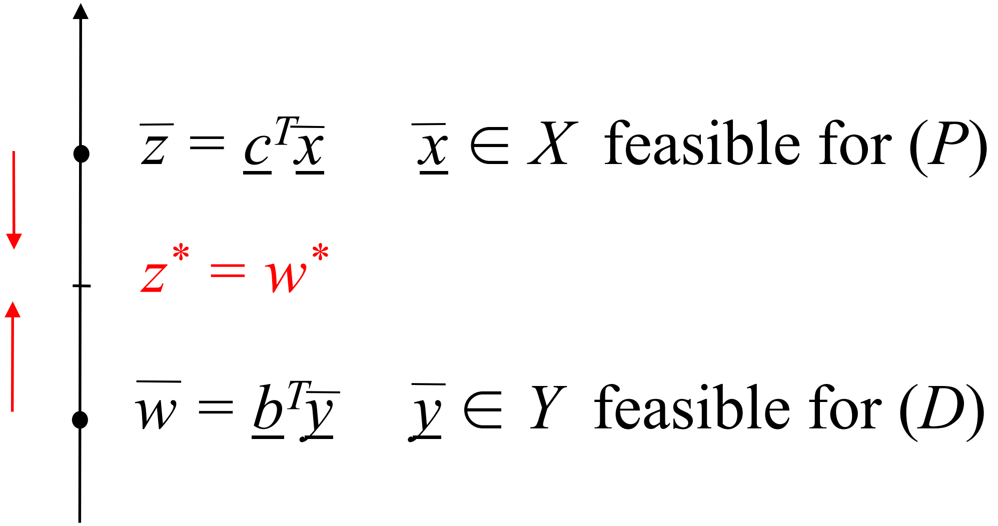
$\underline{\bar{x}}$ is optimal for (P) and $\underline{\bar{y}}$ is optimal for (D).

Optimal solutions are denoted by $\underline{x}^*$ and $\underline{y}^*$

Strong duality theorem

If $X = \{\underline{x} : A\underline{x} \geq \underline{b}\ ,\ \underline{x} \geq \underline{0}\} \neq \emptyset$ and $\min\{\underline{c}^T\underline{x} : \underline{x} \in X\}$ is finite, there exist $\underline{x}^* \in X$ and $\underline{y}^* \in Y$ such that $\underline{c}^T\underline{x}^* = \underline{b}^T\underline{y}^*$.

$$\min\{\ \underline{c}^T\underline{x} : \underline{x} \in X\ \} = \max\{\ \underline{b}^T\underline{y} : \underline{y} \in Y\ \}$$



$\overline{z} = \underline{c}^T\underline{\overline{x}}$ $\quad \underline{\overline{x}} \in X$ feasible for (P)

$z^* = w^*$

$\overline{w} = \underline{b}^T\underline{\overline{y}}$ $\quad \underline{\overline{y}} \in Y$ feasible for (D)

Proof Derive an optimal solution of (*D*) from one of (*P*)

Given

$$(P)\quad \begin{array}{rl} \min & \underline{c}^T\underline{x} \\ & A\underline{x} = \underline{b} \\ & \underline{x} \geq \underline{0} \end{array} \qquad\qquad (D)\quad \begin{array}{rl} \max & \underline{y}^T\underline{b} \\ & \underline{y}^T A \leq \underline{c}^T \\ & \underline{y} \in \mathrm{R}^m \end{array}$$

and $\underline{x}^*$ is an optimal feasible solution of (*P*)

$$\underline{x}^* = \begin{pmatrix} \underline{x}^*_B \\ \underline{x}^*_N \end{pmatrix} \quad \text{with} \quad \begin{cases} \underline{x}^*_B = B^{-1}\underline{b} \\ \underline{x}^*_N = \underline{0} \end{cases}$$

provided (after a finite # of iterations) by the Simplex algorithm with Bland's rule.

Consider $\underline{\bar{y}}^T := \underline{c}^T{}_B B^{-1}$

- Verify that $\underline{\bar{y}}$ is a feasible solution of (D):

$$\underline{\bar{c}}^T{}_N = \underline{c}^T{}_N - (\underline{c}^T{}_B B^{-1})N = \underline{c}^T{}_N - \underline{\bar{y}}^T N \geq \underline{0}^T$$

reduced costs of the nonbasic variables

since $\underline{x}^*$ is optimal

$$\Rightarrow \quad \underline{\bar{y}}^T N \leq \underline{c}^T{}_N$$

$$\underline{\bar{c}}^T{}_B = \underline{c}^T{}_B - (\underline{c}^T{}_B \underbrace{B^{-1})B}_{I} = \underline{c}^T{}_B - \underline{\bar{y}}^T B = \underline{0}^T \Rightarrow \underline{\bar{y}}^T B \leq \underline{c}^T{}_B$$

reduced costs of the basic variables

- According to weak duality, $\underline{\bar{y}}$ is an optimal solution of (D):

$$\underline{\bar{y}}^T \underline{b} = (\underline{c}^T{}_B B^{-1})\underline{b} = \underline{c}^T{}_B (B^{-1}\underline{b}) = \underline{c}^T{}_B \underline{x}^*{}_B = \underline{c}^T \underline{x}^*$$

Hence $\underline{\bar{y}} = \underline{y}^*$

Corollary

For any pair of primal-dual problems (*P*) and (*D*), only four cases can arise:

D *P*	∃ optimal solution	unbounded LP	infeasible LP
∃ optimal solution	~~1)~~	1)	1)
unbounded LP	1)	2)	~~2)~~
Infeasible LP	1)	~~3)~~	~~4)~~

Strong duality theorem ⇒ 1)

Weak duality theorem ⇒ 2) and 3)

4) can arise:

$$(P)\quad \begin{array}{ll} \min & -4x_1 - 2x_2 \\ & -x_1 + x_2 \geq 2 \\ & x_1 - x_2 \geq 1 \\ & x_1, x_2 \geq 0 \end{array} \qquad\qquad (D)\quad \begin{array}{ll} \max & 2y_1 + y_2 \\ & -y_1 + y_2 \leq -4 \\ & y_1 - y_2 \leq -2 \\ & y_1, y_2 \geq 0 \end{array}$$

empty feasible regions

Economic interpretation

The primal and dual problems correspond to two complementary point of views on the same “market”.

Diet problem:

n	aliments	$j=1,\ldots, n$	
m	nutrients	$i=1,\ldots, m$	(vitamines,…)
a_{ij}	quantity of i-th nutrient in one unit of j-th aliment		
b_i	requirement of i-th nutrient		
c_j	cost of one unit of j-th aliment		

$$
(P) \quad
\begin{aligned}
\min \quad & \sum_{j=1}^{n} c_j x_j \\
& \sum_{j=1}^{n} a_{ij} x_j \geq b_i \qquad \forall i = 1,\ldots,m \\
& x_j \geq 0 \qquad \forall j = 1,\ldots,n
\end{aligned}
$$

$$
(D) \quad
\begin{aligned}
\max \quad & \sum_{i=1}^{m} b_i y_i \\
& \sum_{i=1}^{m} a_{ij} y_i \leq c_j \qquad \forall j = 1,\ldots,n \\
& y_i \geq 0 \qquad \forall i = 1,\ldots,m
\end{aligned}
$$

Interpretation of the dual problem:

A company that produces pills of the m nutrients needs to decide the nutrient unit prices y_i so as to maximize income.

- If the costumer buys nutrient pills, he will buy b_i units for each i, $1 \leq i \leq m$.
- The price of the nutrient pills must be competitive:

$$\underbrace{\sum_{i=1}^{m} a_{ij} y_i}_{\text{cost of the pills that are equivalent to 1 unit of } j\text{-th aliment}} \leq c_j \qquad \forall j = 1,\ldots,n$$

If both linear programs (*P*) and (*D*) admit a feasible solution, the strong duality theorem implies that

$$z^* = w^*$$

An "equilibrium" exists (two alternatives with the same cost).

Observation: Strong connection with Game theory (zero-sum games).

Optimality conditions

Given min $z = \underline{c}^T\underline{x}$ max $w = \underline{b}^T\underline{y}$

$$(P) \quad X \begin{cases} A\underline{x} \geq \underline{b} \\ \underline{x} \geq \underline{0} \end{cases} \qquad\qquad (D) \quad Y \begin{cases} \underline{y}^T A \leq \underline{c}^T \\ \underline{y} \geq \underline{0} \end{cases}$$

two feasible solutions $\underline{x}^* \in X$ and $\underline{y}^* \in Y$ are <u>optimal</u>

$$\Leftrightarrow \underline{y}^{*\,T}\underline{b} = \underline{c}^T\underline{x}^*$$

If x_j and y_i are unknown, it is a single equation in $n+m$ unknowns!

Since $\underline{y}^{*T}\underset{\substack{\rotatebox{-90}{$\leq$}\\ A\underline{x}^*}}{\underline{b}} \leq \underbrace{\underline{y}^{*T}A}_{\substack{\rotatebox{-90}{$\leq$}\\ \underline{c}^T}}\underline{x}^* \leq \underline{c}^T\underline{x}^*$, we have

$$\underline{y}^{*T}\underline{b} = \underline{y}^{*T}A\underline{x}^* \quad \text{and} \quad \underline{y}^{*T}A\underline{x}^* = \underline{c}^T\underline{x}^*$$

and therefore

$$\underset{\substack{\rotatebox{90}{$\leq$}\\ \underline{0}^T}}{\underline{y}^{*T}}\underbrace{(A\underline{x}^* - \underline{b})}_{\substack{\rotatebox{90}{$\leq$}\\ \underline{0}}} = 0 \quad \text{and} \quad \underbrace{(\underline{c}^T - \underline{y}^{*T}A)}_{\substack{\rotatebox{90}{$\leq$}\\ \underline{0}^T}}\ \underset{\substack{\rotatebox{90}{$\leq$}\\ \underline{0}}}{\underline{x}^*} = 0$$

$\Rightarrow$ $m+n$ equations in $n+m$ unknowns

<u>necessary</u> and <u>sufficient</u> optimality conditions!

Complementary slackness conditions

$\underline{x}^* \in X$ and $\underline{y}^* \in Y$ are optimal solutions of, respectively, (P) and (D) if and only if

i-th row of *A*

$$y^*_i \, (\overbrace{\underline{a}^T_i \underline{x}^* - b_i}^{\text{slack } s_i \text{ of } i\text{-th constraint of } (P)}) = 0 \qquad \forall i = 1, \ldots, m$$

$$(\underbrace{c^T_j - \underline{y}^{*T} A_j}_{\text{slack } s^D_i \text{ of } j\text{-th constraint of } (D)}) \, x^*_j = 0 \qquad \forall j = 1, \ldots, n$$

j-th column of *A*

At optimality, the product of each variable with the corresponding slack variable of the constraint of the relative dual is = 0.

Economic interpretation for the diet problem

- $\sum_{j=1}^{n} a_{ij} x_j^* > b_i \quad \Rightarrow \quad y_i^* = 0$

If the optimal diet includes an excess of *i*-th nutrient, the costumer is not willing to pay $y^*_i > 0$.

- $y_i^* > 0 \quad \Rightarrow \quad \sum_{j=1}^{n} a_{ij} x_j^* = b_i$

If the company selects a price $y^*_i > 0$, the costumer must not have an excess of *i*-th nutrient.

- $\sum_{i=1}^{m} y_i^* a_{ij} < c_j \Rightarrow x_j^* = 0$

it is not convenient for the costumer to buy aliment *j*

price of the pills equivalent to the nutrients contained in one unit of *j*-th aliment is lower than the price of the aliment.

- $x_j^* > 0 \Rightarrow \sum_{i=1}^{m} y_i^* a_{ij} = c_j$

If costumer includes the *j*-th aliment in optimal diet, the company must have selected competitive prices y^*_i (price of the nutrients in pills contained in a unit of *j*-th aliment is not lower than c_j).

Example:

$$
(P) \quad
\begin{array}{lrl}
\min & 13x_1 + 10x_2 + 6x_3 & \\
s.t. & 5x_1 + x_2 + 3x_3 & = 8 \\
 & 3x_1 + x_2 & = 3 \\
 & x_1, x_2, x_3 & \geq 0
\end{array}
\qquad\qquad
(D) \quad
\begin{array}{lrl}
\max & 8y_1 + 3y_2 & \\
s.t. & 5y_1 + 3y_2 & \leq 13 \\
 & y_1 + y_2 & \leq 10 \\
 & 3y_1 & \leq 6
\end{array}
$$

Verify that the feasible $\underline{x}^* = (1, 0, 1)$ is an optimal, non degenerate solution of (P).

Suppose it is true and derives, via the complementary slackness conditions, the corresponding optimal solution of (D).

Since (P) is in standard form, the conditions

$$y_i^* (\underline{a}^T_i \underline{x}^* - b_i) = 0$$

are automatically satisfied $\forall i,\ 1 \leq i \leq 2$.

Condition $(c^T_j - \underline{y}^{*T} A_j)\, x^*_j = 0$ is satisfied for $j = 2$ because $x^*_2 = 0$.

Since $x^*_1 > 0$ and $x^*_3 > 0$, we obtain the conditions:

$$5y_1 + 3y_2 = 13$$
$$3y_1 = 6$$

and hence the optimal solution $y^*_1 = 2$ and $y^*_2 = 1$ of (D) with $\underline{b}^T \underline{y}^* = 19 = c^T \underline{x}^*$.