

# FOUNDATIONS OF OPERATIONS RESEARCH

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# CHAPTER 1: INTRODUCTION

## Operations Research?

Branch of applied mathematics in which **mathematical models** and **quantitative methods** (e.g., optimization, game theory, simulation) are used to analyze **complex decision-making problems** and find (near-)optimal solutions.

Overall goal: to help make better decisions

Interdisciplinary field at the interface of applied mathematics, computer science, economics and industrial engineering.

## 1.1 Decision-making problems

**Problems** in which we have to choose a (feasible) solution among a large number of alternatives based on one or several criteria.

**Examples:**

### 1) Assignment problem:

Given  $m$  jobs and  $m$  machines, suppose that each job can be executed by any machine and that  $t_{ij}$  is the execution time of job  $J_i$  on machine  $M_j$ .

	$M_1$	$M_2$	$M_3$
$J_1$	2	6	3
$J_2$	8	4	9
$J_3$	5	7	8

$t_{ij}$  matrix ( $m = 3$ )

Decide which job to assign to each machine so as to minimize the total execution time.

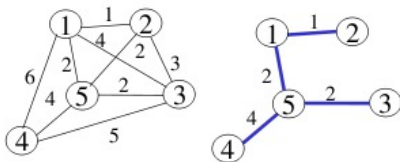
Each job must be assigned to exactly one machine, and each machine to exactly one job.

Number of feasible solutions?  $m!$  possible assignments (permutations)

## 2) Network design:

Decide how to connect  $n$  cities (offices) via a collection of possible links so as to minimize the total link cost.

Given a graph  $G = (N, E)$  with a node  $i \in N$  for each city and an edge  $[i, j] \in E$  of cost  $c_{ij}$  for each link, select a subset of edges of minimum total cost, guaranteeing that all pairs of nodes are connected.



Number of alternative solutions:  $\leq 2^m$  where  $m = |E|$

### 3) Shortest paths:

Given a graph that represents a road network with distances (traveling times) for each arc, determine the shortest (fastest) path between two points (nodes).

A screenshot of the Google Maps interface. The search bar at the top contains the text "tavernero via primo maggio" and "como via valleggio". Below the search bar, there are buttons for "Get directions" and "My places". The main map area shows a blue route starting from Taverno (marked with a green 'A') and ending at Como (marked with a green 'B'). The route passes through several towns including Camnago Villa, Car Rotta, Lipomo, and Urigo. The left sidebar shows suggested routes with their respective distances and travel times. The bottom left corner displays driving directions for the first segment of the route.

Google

Get directions My places

tavernero via primo maggio  
como via valleggio  
Add Destination - Show options  
GET DIRECTIONS

Suggested routes

Route	Distance	Time
Via Otrecolle	7.3 km	14 mins
Via Statale per Lecco	7.1 km	15 mins
Via della Rienza	6.1 km	15 mins

Driving directions to Via Valleggio, 22100 Como CO

Via Primo Maggio  
22038 Tavernero CO

- Head east on Via Primo Maggio/SP37 toward Via Pia

#### 4) **Personnel scheduling:**

Determine the week schedule for the hospital personnel so as to minimize the number of people involved (physicians, nurses,...) while meeting the daily requirements.

#### 5) **Service management:**

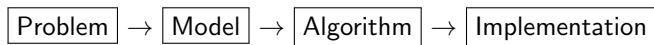
Determine how many counters/desks to open at a given time of the day so that the average customer waiting time does not exceed a certain value (guarantee a given service quality).

#### 6) **Multicriteria problem:**

Decide which laptop to buy considering the price, the weight and the performances.

Complex decision-making problems that are tackled via a **mathematical modelling approach** (mathematical models, algorithms and computer implementations).

## 1.2 Scheme of an O.R. study



Main steps:

- define the problem
- build the model
- select or develop an appropriate algorithm
- implement or use an efficient computer program
- analyze the results

with feedbacks (the previous steps are reconsidered whenever useful).

A mathematical **model** is a simplified representation of a real-world problem.

To define a model we need to identify the fundamental elements of the problem and the main relationships among them.

## 1.3 Historical sketch – part I

Origin in World War II: teams of scientists were asked to do **research** on the most efficient way to conduct the **operations**, e.g., to optimize the allocation of the scarce resources.

Examples: radar location, manage convoy operations, logistics,...

In the decades after the war, the techniques began to be applied more widely to problems in business, industry and society.

During the industrial boom, the substantial increase in the size of the companies and organizations gave rise to more complex decision-making problems.



Favourable circumstances:

- Fast progress in Operations Research and Numerical Analysis methodologies
- Advent and diffusion of computers (available computing power and widespread software).

Operations Research and Management Science are often used as synonyms

## 1.4 Examples of decision-making problems and models

Simplified versions of three problems:

- production planning
- portfolio selection
- facility location

arising in three important fields of application.

## Example 1: Production planning

A company produces three types of electronic devices.

Main phases of the production process: assembly, refinement and quality control

Time in minutes required for each phase and product:

	$D_1$	$D_2$	$D_3$
Assembly	80	70	120
Refinement	70	90	20
Quality control	40	30	20

Available resources within the planning horizon (depend on workforce) in minutes:

Assembly	Refinement	Quality control
30000	25000	18000

Unitary profit expressed in KEuro:

$D_1$	$D_2$	$D_3$
1.6	1	2

Assumption: the company can sell whatever it produces.

**Give a mathematical model** for determining a production "plan" which maximizes the total profit.

# Production planning model

## Decision variables:

$x_j$  = number of devices  $D_j$  produced      for  $j = 1, 2, 3$

## Objective function:

$$\max z = 1.6x_1 + 1x_2 + 2x_3$$

## Constraints: production capacity limit for each phase

$$\begin{aligned} 80x_1 + 70x_2 + 120x_3 &\leq 30000 && \text{(assembly)} \\ 70x_1 + 90x_2 + 20x_3 &\leq 25000 && \text{(refinement)} \\ 40x_1 + 30x_2 + 20x_3 &\leq 18000 && \text{(quality control)} \end{aligned}$$

## Non-negative variables:

$x_1, x_2, x_3 \geq 0$       may take fractional (real) values

## Example 2: Portfolio selection problem

An insurance company must decide which investments to select out of a given set of possible assets (stocks, bonds, options, gold certificates, real estate,...).

Investments	area	capital ( $c_j$ in KEuro)	expected return ( $r_j$ )
A (automotive)	Germany	150	11%
B (automotive)	Italy	150	9%
C (ICT)	U.S.A.	60	13%
D (ICT)	Italy	100	10%
E (real estate)	Italy	125	8%
F (real estate)	France	100	7%
G (short term treasury bonds)	Italy	50	3%
H (long term treasury bonds)	UK	80	5%

Available capital = 600 KEuro

At most 5 investments to avoid excessive fragmentation.

Geographic diversification to limit risk:  $\leq 3$  investments in Italy and  $\leq 3$  abroad.

**Give a mathematical model** for deciding which investments to select so as to maximize the expected return while satisfying the constraints.

# Portfolio selection model

Decision variables:

$x_j = 1$  if  $j$ -th investment is selected and  $x_j = 0$  otherwise for  $j = 1, \dots, 8$

Objective function:

$$\max z = \sum_{j=1}^8 c_j r_j x_j$$

Constraints:

$$\sum_{j=1}^8 c_j x_j \leq 600 \quad (\text{capital})$$

$$\sum_{j=1}^8 x_j \leq 5 \quad (\text{max 5 investments})$$

$$x_2 + x_4 + x_5 + x_7 \leq 3 \quad (\text{max 3 in Italy})$$

$$x_1 + x_3 + x_6 + x_8 \leq 3 \quad (\text{max 3 abroad})$$

Binary (integer) variables:  $x_j \in \{0, 1\} \quad \forall j, 1 \leq j \leq 8$

**Variante:** In order to limit the risk, if any of the ICT investment is selected then at least one of the treasury bond must be selected.

**Model:**

$$\max z = \sum_{j=1}^8 c_j r_j x_j$$

$$\sum_{j=1}^8 c_j x_j \leq 600 \quad (\text{capital})$$

$$\sum_{j=1}^8 x_j \leq 5 \quad (\text{max 5 investments})$$

$$x_2 + x_4 + x_5 + x_7 \leq 3 \quad (\text{max 3 in Italy})$$

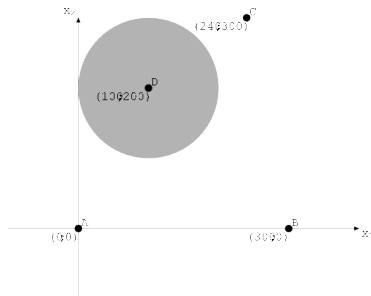
$$x_1 + x_3 + x_6 + x_8 \leq 3 \quad (\text{max 3 abroad})$$

$$\frac{x_3 + x_4}{2} \leq x_7 + x_8 \quad (\text{investment in treasury bonds})$$

$$x_j \in \{0, 1\} \quad \forall j, 1 \leq j \leq 8$$

## Example 3: Facility location

Consider three oil pits, located in positions  $A$ ,  $B$  and  $C$ , from which oil is extracted.



Connect them to a refinery with pipelines whose cost is proportional to the square of their length.

The refinery must be at least 100 km away from point  $D = (100, 200)$ , but the oil pipelines can cross the corresponding forbidden zone.

**Give a mathematical model** to decide where to locate the refinery so as to minimize the total pipeline cost.



# Facility location model

Decision variables:

$x_1, x_2$  cartesian coordinates of the refinery

Objective function:

$$\min z = [(x_1 - 0)^2 + (x_2 - 0)^2] + [(x_1 - 300)^2 + (x_2 - 0)^2] + [(x_1 - 240)^2 + (x_2 - 300)^2]$$

Constraints:

$$\sqrt{(x_1 - 100)^2 + (x_2 - 200)^2} \geq 100$$

Variables:

$$x_1, x_2 \in \mathbb{R}$$

## 1.5 Main features of decision-making problems

- number of decision makers (who decides?)
- number of objectives (based on which criteria?)
- level of uncertainty in the parameters (based on which information?)

You may think of each one of these three features as a dimension of an abstract coordinate system...

One decision maker, one objective  $\Rightarrow$  Mathematical programming

One decision maker, several objectives  $\Rightarrow$  Multi-objective programming

Uncertainty level  $> 0 \Rightarrow$  Stochastic programming

Several decision makers  $\Rightarrow$  Game theory

## 1.6 Mathematical Programming/Optimization (MP)

Decision-making problems with a **single decision maker**, a **single objective** and reliable parameter estimates

$$\text{opt } f(\mathbf{x}) \quad \text{with } \mathbf{x} \in X \quad \text{and} \quad \text{opt} = \left\{ \begin{array}{l} \min \\ \max \end{array} \right\}$$

- **decision variables**  $\mathbf{x} \in \mathbb{R}^n$  : numerical variables whose values identify a solution of the problem
- **feasible region**  $X \subseteq \mathbb{R}^n$  distinguishes between feasible and infeasible solutions (via constraints)

$$X = \left\{ \mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \left\{ \begin{array}{l} = \\ \leq \\ \geq \end{array} \right\} 0, i = 1, \dots, m \right\}$$

- **objective function**  $f : X \rightarrow \mathbb{R}$  expresses in quantitative terms the value or cost of each feasible solution.

Observation:  $\max\{f(\mathbf{x}) : \mathbf{x} \in X\} = -\min\{-f(\mathbf{x}) : \mathbf{x} \in X\}$

# Global optima

Solving a mathematical programming (MP) problem consists in finding a feasible solution which is **globally optimum**, i.e., a vector  $\mathbf{x}^* \in X$  such that

$$\begin{aligned} f(\mathbf{x}^*) &\leq f(\mathbf{x}) & \forall \mathbf{x} \in X & \quad \text{if opt} = \text{min} \\ f(\mathbf{x}^*) &\geq f(\mathbf{x}) & \forall \mathbf{x} \in X & \quad \text{if opt} = \text{max.} \end{aligned}$$

It may happen that:

- 1 the problem is infeasible ( $X = \emptyset$ ),
- 2 the problem is unbounded ( $\forall c \in \mathbb{R}, \exists \mathbf{x}_c \in X$  such that  $f(\mathbf{x}_c) \leq c$  or  $f(\mathbf{x}_c) \geq c$ ),
- 3 the problem has a single optimal solution,
- 4 the problem has a large number (even an infinite number) of optimal solutions (with the same optimal value!).

## Local optima

When the problem at hand is very hard we must settle for a feasible solution that is a **local optimum**, i.e., a vector  $\hat{\mathbf{x}} \in X$  such that

$$\begin{aligned} f(\hat{\mathbf{x}}) &\leq f(\mathbf{x}) && \forall \mathbf{x} \text{ with } \mathbf{x} \in X \text{ and } \|\mathbf{x} - \hat{\mathbf{x}}\| \leq \epsilon && \text{if opt} = \text{min} \\ f(\hat{\mathbf{x}}) &\geq f(\mathbf{x}) && \forall \mathbf{x} \text{ with } \mathbf{x} \in X \text{ and } \|\mathbf{x} - \hat{\mathbf{x}}\| \leq \epsilon && \text{if opt} = \text{max} \end{aligned}$$

for an appropriate value  $\epsilon > 0$ .

An optimization problem can have many local optima.

# Special cases of Mathematical Programming

- **Linear Programming (LP)**

$f(\mathbf{x})$  linear

$$X = \left\{ \mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \begin{cases} = \\ \leq \\ \geq \end{cases} 0, i = 1, \dots, m \right\} \text{ with } g_i(\mathbf{x}) \text{ linear for each } i$$

Example: Production planning

- **Integer Linear Programming (ILP)**

$f(\mathbf{x})$  linear

$$X = \left\{ \mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \begin{cases} = \\ \leq \\ \geq \end{cases} 0, i = 1, \dots, m \right\} \cap \mathbb{Z}^n \text{ with } g_i(\mathbf{x}) \text{ linear } \forall i$$

Examples: Portfolio selection

ILP coincides with LP with the additional integrality constraint on the variables

- **Nonlinear Programming (NLP)**

$f(\mathbf{x})$  convex/regular or non convex/regular

$$X = \left\{ \mathbf{x} \in \mathbb{R}^n : g_i(\mathbf{x}) \begin{cases} = \\ \leq \\ \geq \end{cases} 0, i = 1, \dots, m \right\} \text{ with for each } i \ g_i(\mathbf{x})$$

convex/regular or non convex/regular

Example: Facility location (with  $g_i$  convex)

# Historical sketch: Mathematical Programming

1826/1827: **Joseph Fourier** presents a method to solve systems of linear inequalities (Fourier-Motzkin) and discusses some LPs with 2-3 variables.

1939: **Leonid Kantorovitch** lays the bases of Linear Programming (Nobel prize 1975).

1947: **George Dantzig** proposes independently LP and invents the Simplex algorithm.

1958: **Ralph Gomory** proposes a cutting plane method for ILP problems.



Joseph Fourier  
(1768-1830)



L.V. Kantorovitch  
(1912-1986)



G.B. Dantzig  
(1914-2005)



R.E. Gomory  
(1929-)



## 1.6 Multi-objective programming

Multiple objectives can be taken into account in different ways

Suppose we wish to minimize  $f_1(\mathbf{x})$  and maximize  $f_2(\mathbf{x})$  (laptop example:  $f_1$  is cost and  $f_2$  is performance)

- 1 Turn it into a **single objective problem** by expressing the two objectives in terms of the same unit (e.g., monetary unit)

$$\min \lambda_1 f_1(\mathbf{x}) - \lambda_2 f_2(\mathbf{x})$$

for appropriate scalars  $\lambda_1$  and  $\lambda_2$ .

- 2 Optimize the **primary objective** function and turn the other objective into a constraint

$$\max_{\mathbf{x} \in \tilde{X}} f_2(\mathbf{x}) \quad \text{where } \tilde{X} = \{\mathbf{x} \in X : f_1(\mathbf{x}) \leq c\}$$

for an appropriate constant  $c$ .

## 1.7 Mathematical programming versus simulation

Both approaches involve constructing a **model** and designing an **algorithm**.

### Mathematical programming

Problem can be “well” formalized

Algorithm yields a(n optimal) solution

The results are “certain”

Example: assignment

### Simulation

Problem is difficult to formalize

Algorithm simulates the evolution of the real system and allows to evaluate the performance of alternative solutions.

The results need to be interpreted

Example: service counters

## 1.8 The impact of Operations Research

year	company	sector	results
90	Taco Bell (fast food)	personnel scheduling	7.6M\$ annual savings
92	American Airlines	design fare structure, overbooking and flights coordination	+ 500M\$
92	Harris Corp. (semiconductors)	production planning	50% $\Rightarrow$ 95% orders on time
95	GM - car rental	use of car park	+50M\$ per year avoided bankruptcy
96	HP - printers	modify production line	doubled production
97	Bosques Arauco	harvesting logistics and transport	5M\$ annual savings
99	IBM	supply chain re-engineering	750M\$ annual savings

Most of the top managers interviewed by "Fortune 500" declared that they used O.R. methodologies.

**Video:** <http://www.bnet.com/videos/operations-research-critical-applications-for-business/178846>



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## Success Stories

We've put together a library of dozens of brief O.R. cases for you to browse or search by industry, functional area, or benefit.

A few examples of the value you'll find:

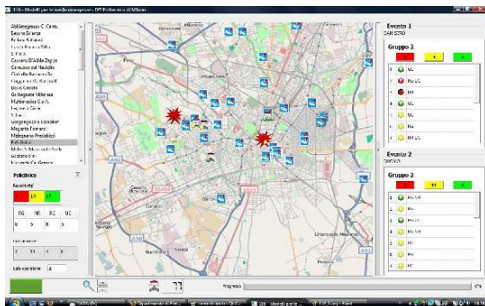
- **Ford used O.R.** to optimize the way it designs and tests vehicle prototypes, saving \$250 million.
- **UPS used O.R.** to redesign its overnight delivery network, saving \$87 million plus an additional \$189 million over the following decade.
- **The U.S. Army used O.R.** to increase recruiting 17.5% while saving 20%.
- **NBC used O.R.** to improve advertising sales plans, increasing revenues by over \$200 million.
- The City of New Haven used O.R. to determine definitively that its needle exchange program was reducing HIV infection rates.

Sort by criteria

Industry	<input type="text" value="Select an industry"/>
Function	<input type="text" value="Select a function"/>
Benefit	<input type="text" value="Select a benefit"/>

<http://www.scienceofbetter.org/>

## Not just for money. Example: a regional project with Milan Emergency Service 118



Significant impact not only for large companies and organizations.

In rapidly evolving contexts, characterized by strong competitiveness, high levels of complexity and uncertainty, it is crucial to identify and implement efficient solutions.

The huge amount of data available with modern information systems (Big data) opens new avenues...