5.3 Cutting plane methods and Gomory fractional cuts

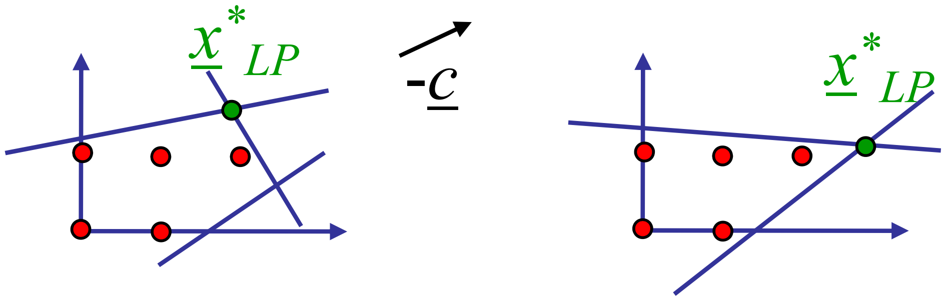
$$\text{(ILP)} \quad \begin{array}{ll} \min & \underline{c}^T\underline{x} \\ s.t. & A\underline{x} \geq \underline{b} \\ & \underline{x} \geq \underline{0} \ \text{ integer} \end{array} \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{ feasible region } X$$

Assumption: a_{ij}, c_j and b_i integer.

Observation: The feasible region of an ILP can be described by different sets of constraints that may be weaker/tighter.

infinitely many formulations!

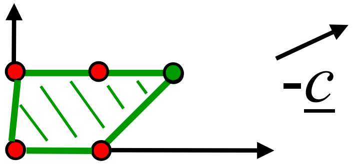
Equivalent and ideal formulations



∞ formulations

All formulations (with integrality constraints) are equivalent but the optimal solutions of the linear relaxations ($\underline{x}^*_{LP}$) can differ substantially.

Definition: The *ideal formulation* is that describing the convex hull *conv*(*X*) of the feasible region *X*, where *conv*(*X*) is the smallest convex subset containing *X*.



Since all vertices have all integer coordinates, for any $\underline{c}$ we have $z^*_{LP} = z^*_{ILP}$ and LP optimum is also ILP optimum !

bounded or unbounded

Theorem: For any feasible region X of an ILP, there exists an ideal formulation (a description of $conv(X)$ involving a finite number of linear constraints) but the number of constraints can be very large (exponential) with respect to the size of the original formulation.

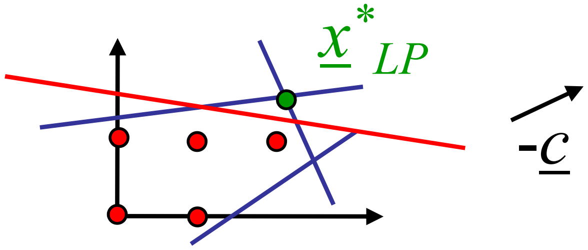
In theory, the solution of any ILP can be reduced to that of a single LP!

However, the ideal formulation is often either very large and/or very difficult to determine.

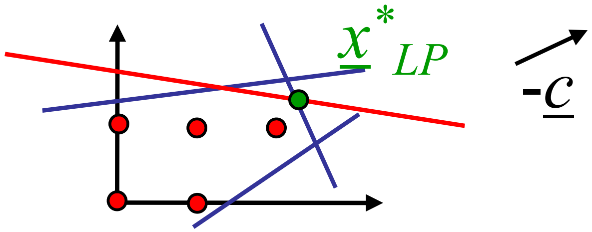
5.3.1 Cutting plane methods

A full description of *conv*(X) is not required, we just need a good description in the neighborhood of the optimal solution.

Definition: A cutting plane is an inequality $\underline{a}^T\underline{x} \leq b$ that is not satisfied by $\underline{x}^*_{LP}$ but is satisfied by all the feasible solutions of the ILP.



Idea: Given an initial formulation, iteratively add cutting planes as long as the linear relaxation does not provide an optimal integer solution.



etc...

5.3.2. Gomory fractional cuts

Ralph Gomory 1929-

Let $\underline{x}^*_{LP}$ be an optimal solution of the linear relaxation of the current formulation $\min\{\underline{c}^T\underline{x} : A\underline{x} = \underline{b},\ \underline{x} \geq \underline{0}\}$ and $x^*_{B[r]}$ be a fractional basic variable.

The corresponding row of the optimal tableau:

fractional

$$x_{B[r]} + \sum_{j : x_j \in N} \bar{a}_{rj}\, x_j = \bar{b}_r \qquad (*)$$

x_j non basic

Definition: *Gomory cut* w.r.t. the fractional basic variable $x_{B[r]}$:

$$\sum_{j : x_j \in N} (\bar{a}_{rj} - \lfloor \bar{a}_{rj} \rfloor)\, x_j \geq (\bar{b}_r - \lfloor \bar{b}_r \rfloor)$$

Let us verify that the inequality

$$\sum_{j : x_j \in N} (\bar{a}_{rj} - \lfloor \bar{a}_{rj} \rfloor)\, x_j \geq (\bar{b}_r - \lfloor \bar{b}_r \rfloor)$$

is a cutting plane with respect to $\underline{x}^*_{LP}$.

- It is violated by the optimal fractional solution $\underline{x}^*_{LP}$ of the linear relaxation:

  Obvious since $(\bar{b}_r - \lfloor \bar{b}_r \rfloor) > 0$ and $x_j = 0$ $\forall j$ s.t. x_j non basic.

- It is satified by all integer feasible solution:

For each feasible solution of the linear relaxation, we have

$$x_{B[r]} + \sum_{j \in N} \lfloor \bar{a}_{rj} \rfloor \, x_j \leq x_{B[r]} + \sum_{j \in N} \bar{a}_{rj} \, x_j = \bar{b}_r \qquad x_j \geq 0$$

and, in particular, for each integer feasible solution

$$x_{B[r]} + \sum_{j \in N} \lfloor \bar{a}_{rj} \rfloor \, x_j \leq \lfloor \bar{b}_r \rfloor \qquad (**) \qquad x_j \text{ integer}$$

By substracting (**) from (*), for each integer feasible solution we have:

$$\sum_{j \in N} (\bar{a}_{rj} - \lfloor \bar{a}_{rj} \rfloor) \, x_j \geq (\bar{b}_r - \lfloor \bar{b}_r \rfloor).$$

The “integer” form

$$x_{B[r]} + \sum_{j \in N} \lfloor \bar{a}_{rj} \rfloor \, x_j \leq \lfloor \bar{b}_r \rfloor$$

and the “fractional” form

$$\sum_{j \in N} (\bar{a}_{rj} - \lfloor \bar{a}_{rj} \rfloor) \, x_j \geq (\bar{b}_r - \lfloor \bar{b}_r \rfloor)$$

of the cutting plane are obviously equivalent.

Example:

$$
\begin{array}{llcl}
\max & z = 8x_1 + 5x_2 & & \\
 & x_1 \quad + x_2 & \leq & 6 \\
 & 9x_1 \quad + 5x_2 & \leq & 45 \\
 & \qquad x_1, x_2 \geq 0 & & \text{integer}
\end{array}
$$

slack
variables

Optimal tableau:

		x_1	x_2	s_1	s_2
$-z$	-41.25	0	0	-1.25	-0.75
x_1	3.75	1	0	-1.25	0.25
x_2	2.25	0	1	2.25	-0.25

with the fractional optimal basic solution $\underline{x}^*_B = \begin{pmatrix} 3.75 \\ 2.25 \end{pmatrix}$

Select a row of the optimal tableau (a constraint) whose basic variable has a fractional value:

$$x_1 - 1.25\, s_1 + 0.25\, s_2 = 3.75$$

Generate the corresponding Gomory cut: $0.75\, s_1 + 0.25\, s_2 \geq 0.75$

Note: The integer and fractional parts of a real number a are

$$a = \lfloor a \rfloor + f \qquad \text{with } 0 \leq f < 1$$

thus we have $-1.25 = -2 + 0.75$ and $0.25 = 0 + 0.25$.

Introduce the slack variable $s_3 \geq 0$ and add this cutting plane to the tableau:

		x_1	x_2	s_1	s_2	s_3
$-z$	-41.25	0	0	-1.25	-0.75	0
x_1	3.75	1	0	-1.25	0.25	0
x_2	2.25	0	1	2.25	-0.25	0
s_3	-0.75	0	0	-0.75	-0.25	1

$$\Leftarrow \; -0.75s_1 - 0.25s_2 \leq -0.75$$

The new constraint "cuts" the optimal fractional solution $\underline{x}^*_B = \begin{pmatrix}3.75\\2.25\end{pmatrix}$ of the linear relaxation of the ILP.

To efficiently reoptimize, we can apply a single iteration of the Dual simplex algorithm.

Optimal tableau:

		x_1	x_2	S_1	s_2	s_3
$-z$	-40	0	0	0	-0.33	-1.67
x_1	5	1	0	0	0.67	-1.67
x_2	0	0	1	0	-1	3
s_1	1	0	0	1	0.33	-1.33

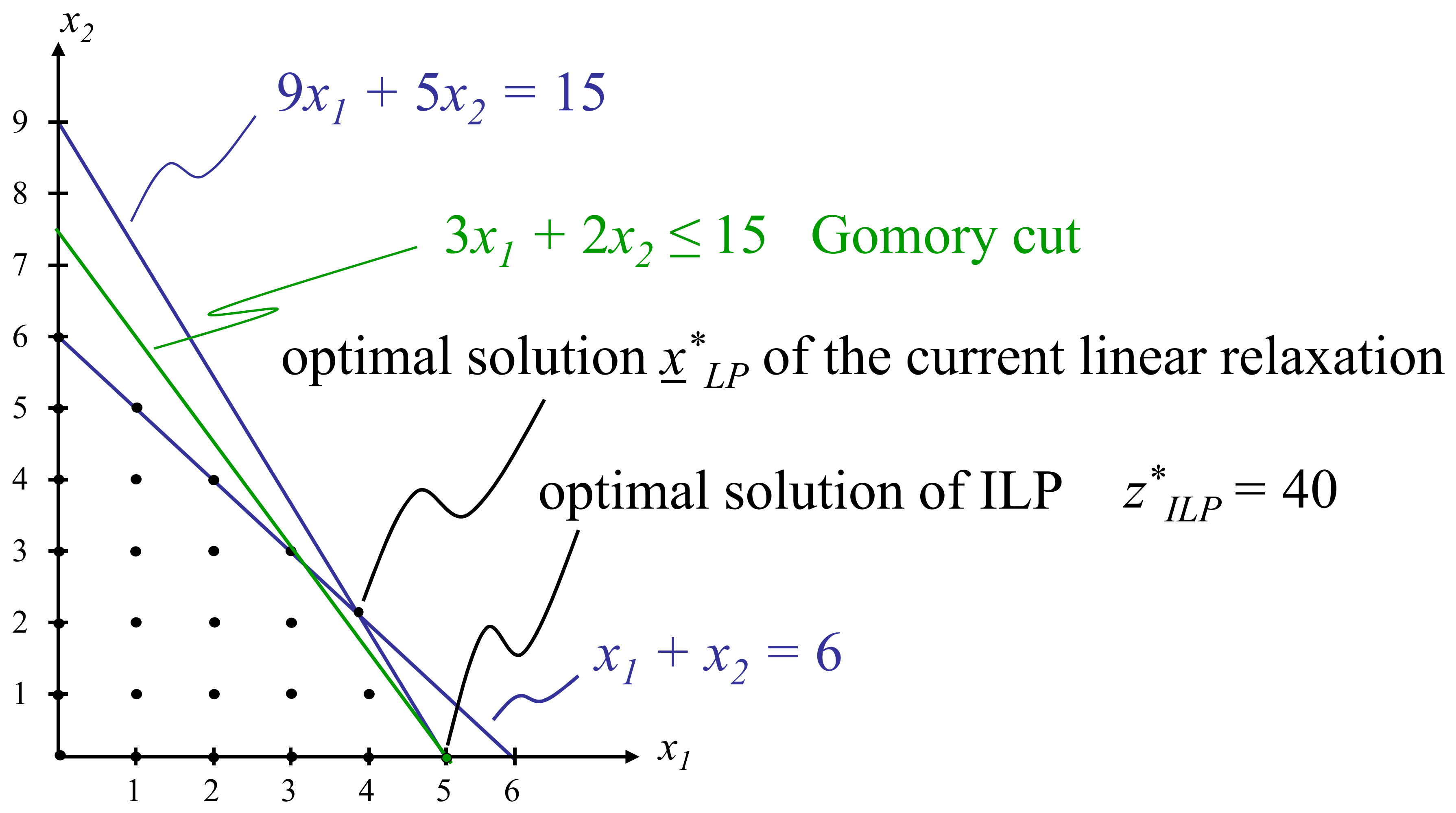
Since the optimal solution $x^*=[5, 0, 1,0, 0]^T$ (with $z^* = 40$) of the linear relaxation of the new formulation is integer, x^* is also optimal for the original ILP and we do not need to generate other Gomory cuts!

To express the Gomory cut

$$0.75\, s_1 + 0.25\, s_2 \geq 0.75$$

In terms of the decision variables, we perform the substitution:

$$\begin{cases} s_1 = 6 - x_1 - x_2 \\ s_2 = 45 - 9x_1 - 5x_2 \end{cases} \quad \Rightarrow \quad 3x_1 + 2x_2 \leq 15$$



$9x_1 + 5x_2 = 15$

$3x_1 + 2x_2 \leq 15$ Gomory cut

optimal solution $\underline{x}^*_{LP}$ of the current linear relaxation

optimal solution of ILP $z^*_{ILP} = 40$

$x_1 + x_2 = 6$

Very special case: original constraints + cut ≡ <u>ideal formulation</u>!

In general we need to add a (very) large number of cuts.

5.3.3 Cutting plane method with fractional Gomory cuts

```
BEGIN
  Solve the linear relaxation $\min\{\underline{c}^T\underline{x} : A\underline{x} = \underline{b},\ \underline{x} \geq \underline{0}\}$
  and let $\underline{x}^*_{LP}$ be an optimal basic feasible solution;
  WHILE $\underline{x}^*_{LP}$ has fractional components DO
    Select a basic variable with a fractional value;
    Generate the corresponding Gomory cut;
    Add the constraint to the optimal tableau of the
    linear relaxation;
    Perform one iteration of the dual simplex algorithm;
  END-WHILE
END
```

Theorem: If the ILP has a finite optimal solution, the cutting plane method finds one after adding a finite number of Gomory cuts.

but often very large

Example:

$$
\text{(ILP)} \qquad
\begin{array}{lrcl}
\min & & -x_2 & \\
 & 3x_1 & +\, 2x_2 & \leq\ 6 \\
 & -3x_1 & +\, 2x_2 & \leq\ 0 \\
 & & x_1, x_2 \geq 0 & \text{integer}
\end{array}
$$

Solve the linear relaxation with the simplex algorithm:

		x_1	x_2	x_3	x_4
$-z$	0	0	-1	0	0
x_3	6	3	2	1	0
x_4	0	-3	②	0	1

$$x_3 = 6 - 3x_1 - 2x_2$$
$$x_4 = 3x_1 - 2x_2$$

$$\begin{array}{c|c|cccc|} & & x_1 & x_2 & x_3 & x_4 \\ \hline -z & 0 & -3/2 & 0 & 0 & ½ \\ \hline x_3 & 6 & \textcircled{6} & 0 & 1 & -1 \\ x_2 & 0 & -3/2 & 1 & 0 & 1/2 \\ \hline \end{array} \qquad \begin{array}{c|c|cccc|} & & x_1 & x_2 & x_3 & x_4 \\ \hline -z & 3/2 & 0 & 0 & ¼ & ¼ \\ \hline x_1 & 1 & 1 & 0 & 1/6 & -1/6 \\ x_2 & 3/2 & 0 & 1 & ¼ & ¼ \\ \hline \end{array}$$

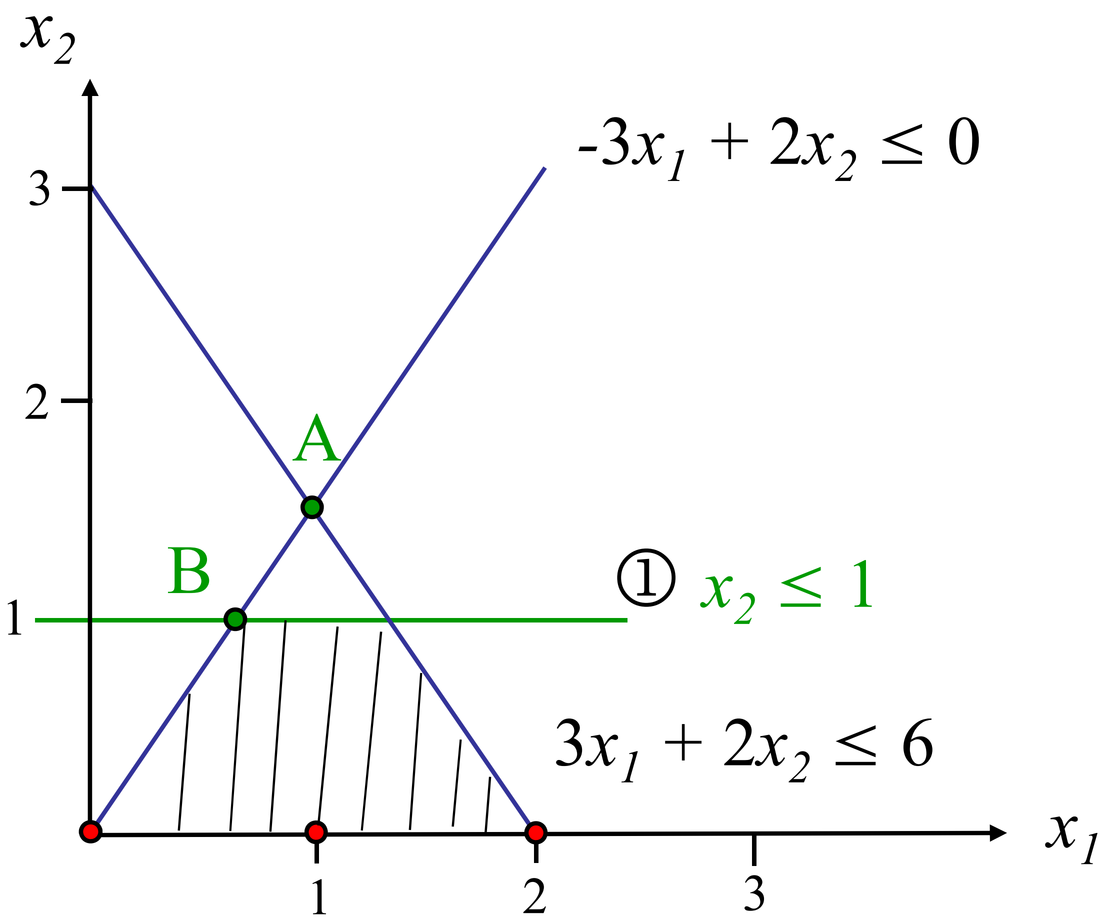
The optimal soluton $x^*=[1, 3/2, 0, 0]^T$ has value $z^*_{LP}= -3/2$ (vertex A).

Generate the Gomory cut associated to the 2nd row:

$$x_2 + ¼\, x_3 + ¼\, x_4 = 3/2 \quad \Rightarrow \quad x_2 + 0x_3 + 0x_4 \leq \lfloor 3/2 \rfloor$$

namely the constraint $x_2 \leq 1$ (cut ①).

Adding to the fractional form $¼\, x_3 + ¼\, x_4 \geq ½$ the surplus variable $x_5 \geq 0$, we obtain: $- ¼\, x_3 – ¼\, x_4 + x_5 = - ½$.



A = (1, 3/2)

B = (2/3, 1)

Graphical representation

Adding the corresponding row to the tableau:

$$x_5 = \text{-}1/2 + ¼\, x_3 + ¼\, x_4$$
$$= \text{-}1/2 \;+ ¼\; (6 – 3x_1 – 2x_2)\;\; ¼\, (3x_1 – 2x_2)$$
$$=1 – x_2$$

		x_1	x_2	x_3	x_4	x_5
$-z$	3/2	0	0	¼	¼	0
x_1	1	1	0	1/6	-1/6	0
x_2	3/2	0	1	¼	¼	0
x_5	-1/2	0	0	-1/4	-1/4	1

In order to represent the cut in the space of the original variables, we proceed by substitution: the new surplus variable x_5 is expressed in terms of only x_1 and x_2.

We obtain the new optimal tableau:

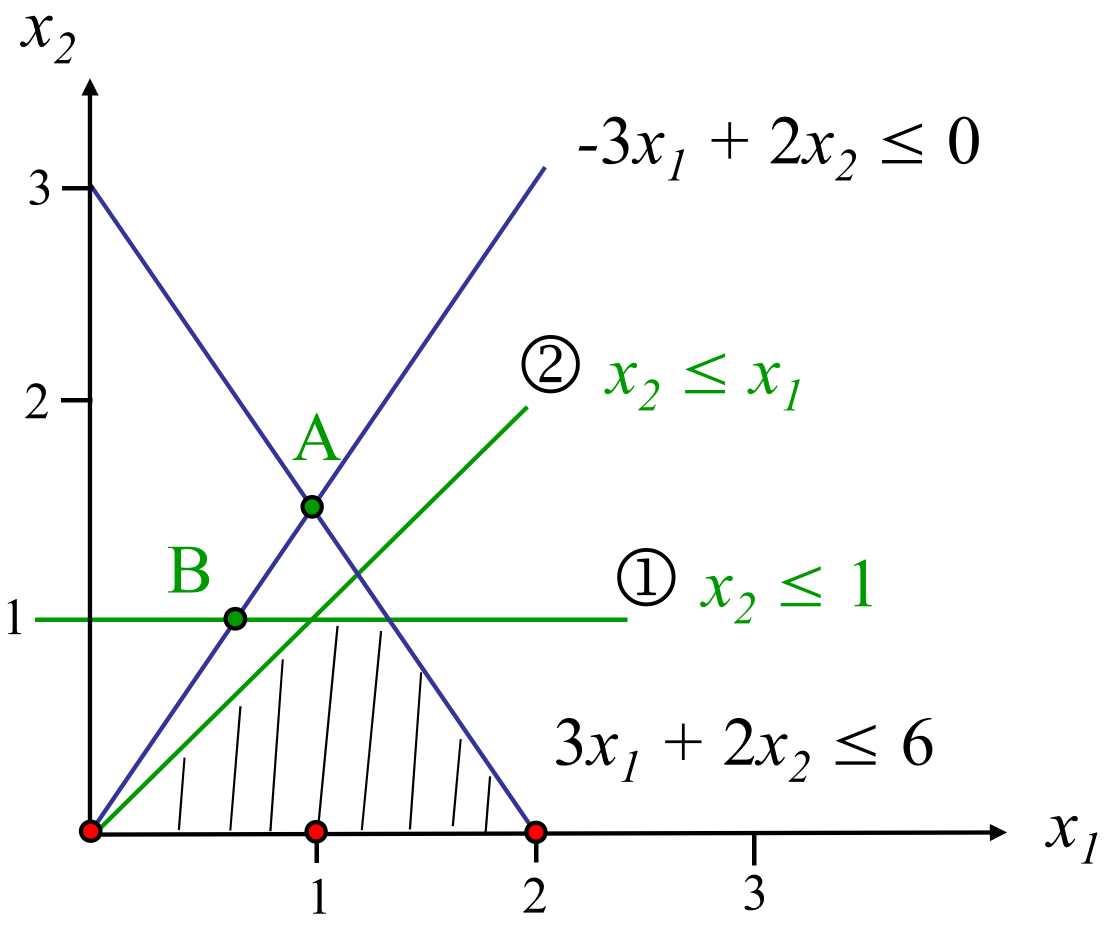
		x_1	x_2	x_3	x_4	x_5
$-z$	1	0	0	0	0	1
x_1	2/3	1	0	0	-1/3	2/3
x_2	1	0	1	0	0	1
x_3	2	0	0	1	1	-4

The optimal solution $x^* = [2/3, 1, 2, 0, 0]^T$ is still fractional (vertex B).

The integer form of the Gomory cut associated to the 1st row is

$x_1 - x_4 \leq \lfloor 2/3 \rfloor = 0$, which by substiting x_4 with $x_4 = 3x_1 - 2x_2$ is equivalent

to $-2x_1 + 2x_2 \leq 0$ (cut ②).



A = (1, 3/2)

B = (2/3, 1)

Graphical representation

Since the fractional form of the cut is $2/3x_4 + 2/3x_5 \geq 2/3$, it suffices to include the surplus variable $x_6 \geq 0$ and add the corresponding row to the "extended" tableau:

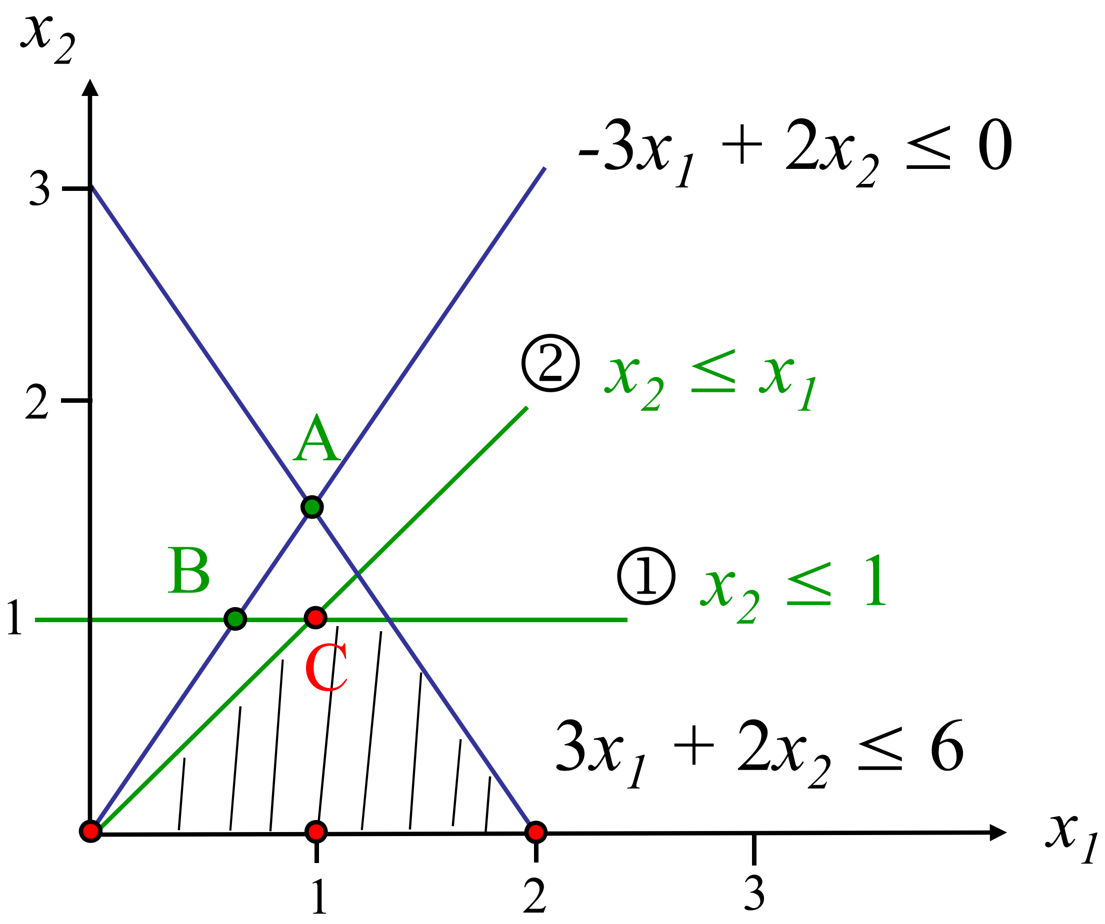
		x_1	x_2	x_3	x_4	x_5	x_6
$-z$	1	0	0	0	0	1	0
x_1	2/3	1	0	0	-1/3	2/3	0
x_2	1	0	1	0	0	1	0
x_3	2	0	0	1	1	-4	0
x_6	-2/3	0	0	0	-2/3	-2/3	1

$$\begin{aligned} x_6 &= -2/3 + 2/3x_4 + 2/3x_5 \\ &= -2/3(3\,x_1 - 2x_2) \\ &\quad + 2/3(1 - x_2) \\ &= 2x_1 - 2x_2 \end{aligned}$$

We obtain the optimal tableau:

$$\begin{array}{c|c|cccccc|}
 & & x_1 & x_2 & x_3 & X_4 & X_5 & x_6 \\
\hline
-z & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
x_1 & 1 & 1 & 0 & 0 & 0 & 1 & -1/2 \\
x_2 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
x_3 & 1 & 0 & 0 & 1 & 0 & -5 & 3/2 \\
x_4 & 1 & 0 & 0 & 0 & 1 & 1 & -3/2 \\
\hline
\end{array}$$

The optimal solution of the linear relaxation $x^* = [1, 1, 1, 1, 0, 0]^T$ corresponds to the vertex C whose components are all integer.



A = (1, 3/2)

B = (2/3, 1)

C = (1, 1)

Graphical representation

Observation:

The formulation is not ideal (the polytope has still a fractional vertex), the constraint $x_1 + x_2 \leq 2$ that is needed to describe $conv(X)$ is not required for this objective function.

5.3.4 Generic and specific cutting planes

There exist other types of generic cutting planes (different from the fractional Gomory cuts) and a large number of classes of cutting planes for specific problems.

The "deepest" cuts are the "facets" of *conv*(*X*) !

The thorough study of the combinatorial structure of various problems (e.g., TSP, set covering, set packing,…) led to

- characterization of entire classes of facets,
- efficient procedures for generating them.

5.3.5 Idea of Branch and Cut

The “combined” Branch and Cut approach aims at overcoming the disadvantages of pure Branch-and-Bound (B&B) and pure cutting plane methods.

For each subproblem (node) of B&B, several cutting planes are generated to improve the bound and try to find an optimal integer solution. Whenever the cutting planes become less effective, cut generation is stopped and a branching operation is performed.

Advantages: The cuts tend to strengthen the formulation (linear relaxation) of the various subproblems; the long series of cuts without sensible improvement are interrupted by branching operations.