5.1 Branch and Bound method

Consider a generic optimization problem

$$\min\{ c(\underline{x}) : \underline{x} \in X \}$$

Idea: Reduce the solution of a difficult problem to that of a sequence of simpler subproblems by (recursive) partition of the feasible region X.

Applicable to discrete and continuous optimization problems.

Two main components: branching and bounding.

$$z = \min\{\, c(\underline{x}) : \underline{x} \in X \,\}$$

Branching:

Partition X into k subsets

$$X = X_1 \cup \ldots \cup X_k \text{ (with } X_i \cap X_j = \emptyset \text{ for each pair } i \neq j \text{)}$$

and let

$$z_i = \min\{\, c(\underline{x}) : \underline{x} \in X_i \,\} \quad \text{for } i = 1,\ldots, k.$$

Clearly $\quad z = \min\{\, c(\underline{x}) : \underline{x} \in X \,\} = \min\{z_1,\ldots, z_k\}$

Bounding technique:

For each subproblem $z_i = \min\{ c(\underline{x}) : \underline{x} \in X_i \}$

i) determine an optimal solution of $\min\{ c(\underline{x}) : \underline{x} \in X_i \}$ (explicit), or

ii) prove that $X_i = \emptyset$ (explicit), or

iii) prove that $z_i \geq z'$ = objective function value of the best feasible solution found so far (implicit)

If the subproblem is not "solved" we generate new subproblems by further partition.

5.1.1 Branch and Bound for ILP

Given an ILP $\min\{ \underline{c}^T \underline{x} : A\underline{x} = \underline{b}, \underline{x} \geq \underline{0} \text{ integer} \}$

Branching:

Partition the feasible region X into subregions (subdivision in exhaustive and exclusive subregions).

Achieved by solving the linear relaxation of the ILP

$$\min\{ \underline{c}^T \underline{x} : A\underline{x} = \underline{b}, \underline{x} \geq \underline{0} \}$$

denote by $\overline{\underline{x}}$ an optimal solution and $z_{LP} = \underline{c}^T \overline{\underline{x}}$ the optimal value.

If $\underline{\bar{x}}$ integer, $\underline{\bar{x}}$ is also optimal for ILP, otherwise

$\exists\, \bar{x}_h$ fractional and we consider the two subproblems:

$$\text{ILP}_1: \quad \min\{ \underline{c}^T \underline{x} : A\underline{x} = \underline{b},\ x_h \leq \lfloor \bar{x}_h \rfloor,\ \underline{x} \geq \underline{0} \text{ integer } \}$$

$$\text{ILP}_2: \quad \min\{ \underline{c}^T \underline{x} : A\underline{x} = \underline{b},\ x_h \geq \lfloor \bar{x}_h \rfloor + 1,\ \underline{x} \geq \underline{0} \text{ integer } \}$$

Bounding:

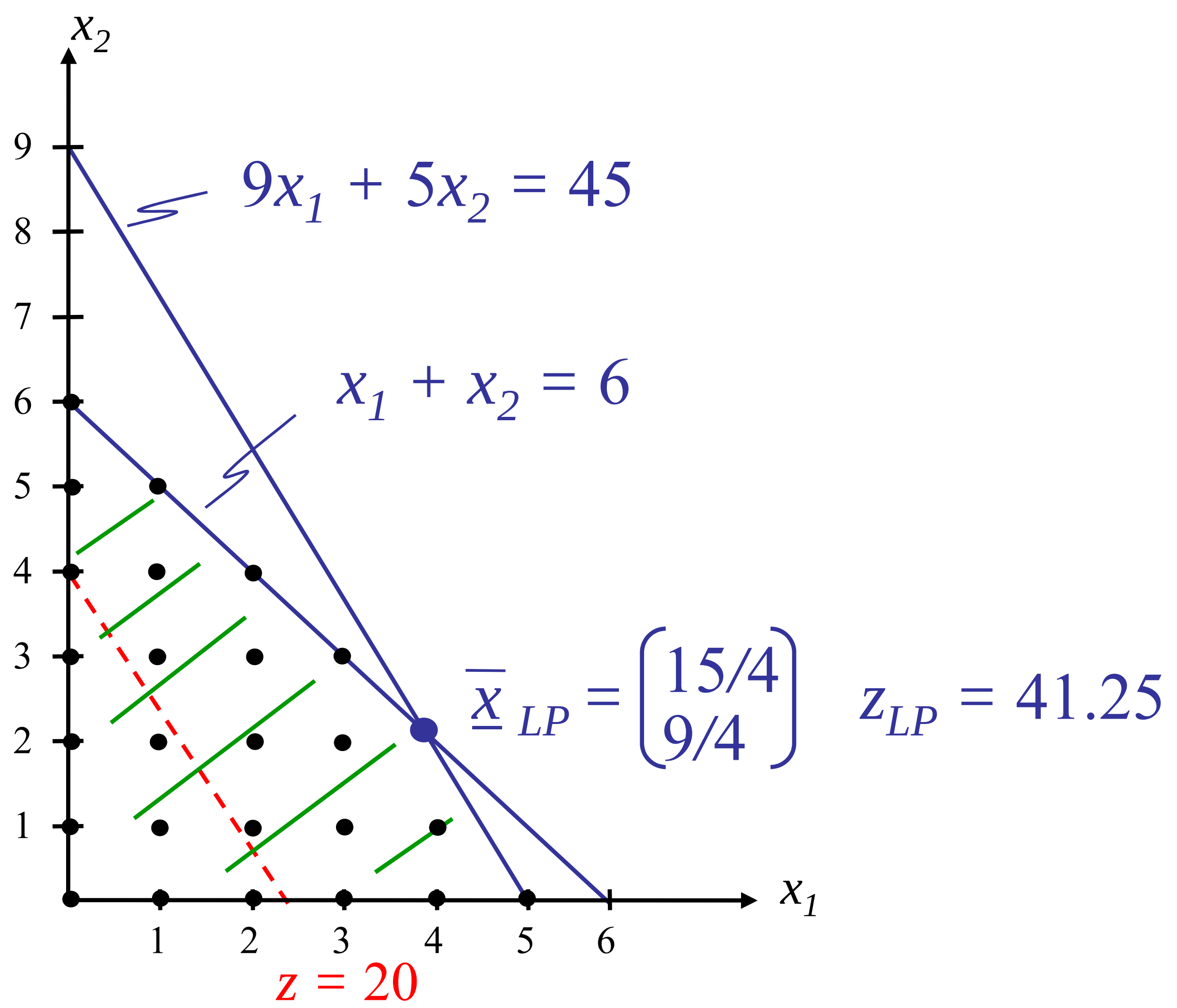
Determine a lower "bound" (if minimization ILP) on the optimal value z_i of a subproblem of ILP by solving its linear relaxation.

Example:

$$\max \quad z = 8x_1 + 5x_2$$

$$\text{(ILP)} \quad \begin{aligned} x_1 + x_2 &\leq 6 \\ 9x_1 + 5x_2 &\leq 45 \\ x_1, x_2 &\geq 0 \text{ integer} \end{aligned}$$

$$z_{LP} \geq z_{ILP}$$



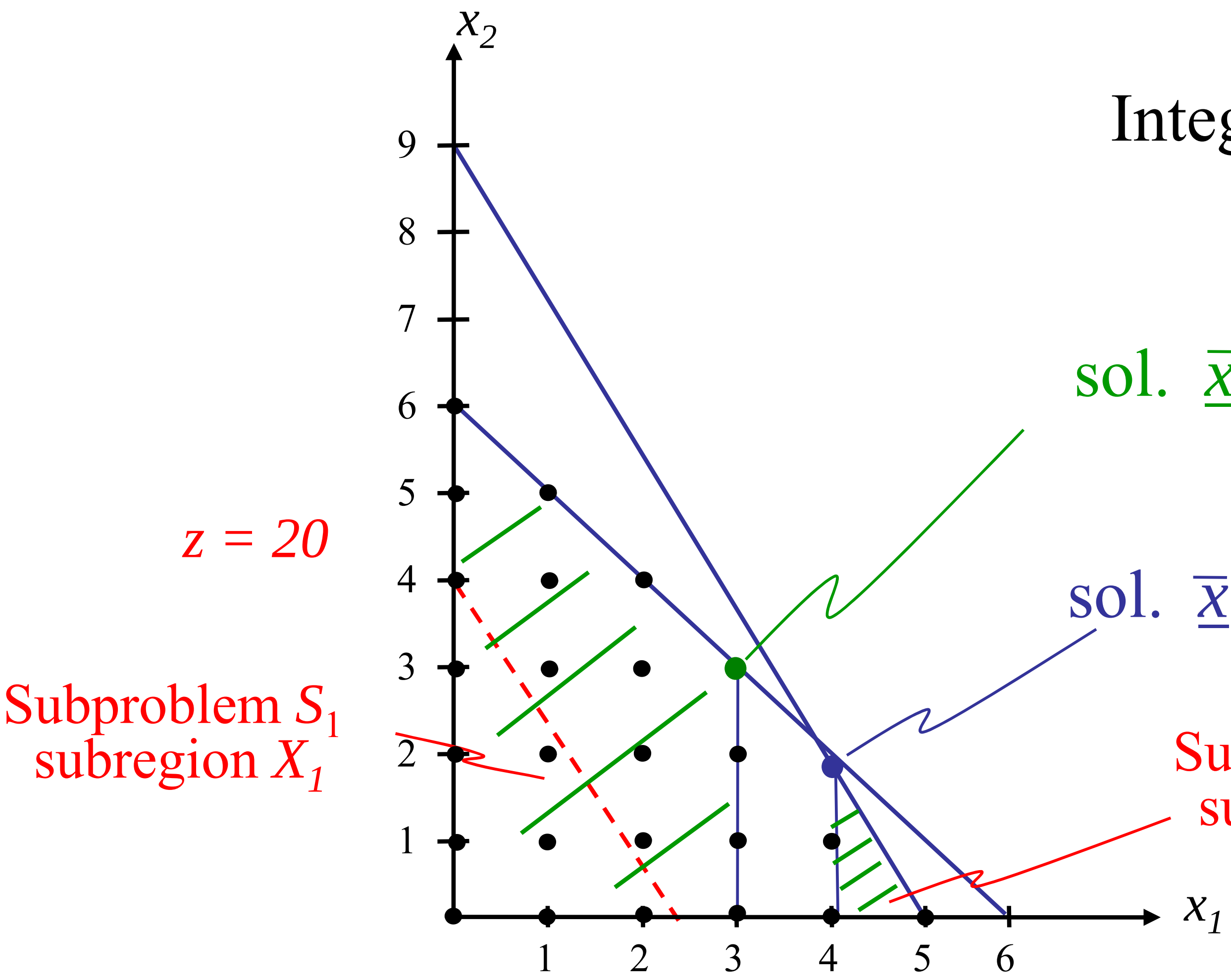
Since $\overline{x}_1$ and $\overline{x}_2$ are fractional, select one of them for branching.

for instance x_1

The feasible region X is partitioned into X_1 and X_2 by imposing:

$$x_1 \leq \lfloor \bar{x}_1 \rfloor = 3 \quad \text{or} \quad x_1 \geq \lfloor \bar{x}_1 \rfloor + 1 = 4$$

<u>exhaustive</u> and <u>exclusive</u> constraints



Integer solution ! $\Rightarrow z_{ILP1} = z_{LP1}$

sol. $\underline{\bar{x}}_{LP1} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ with $z_{LP1} = 39$

sol. $\underline{\bar{x}}_{LP2} = \begin{bmatrix} 4 \\ 9/5 \end{bmatrix}$ with $z_{LP2} = 41$

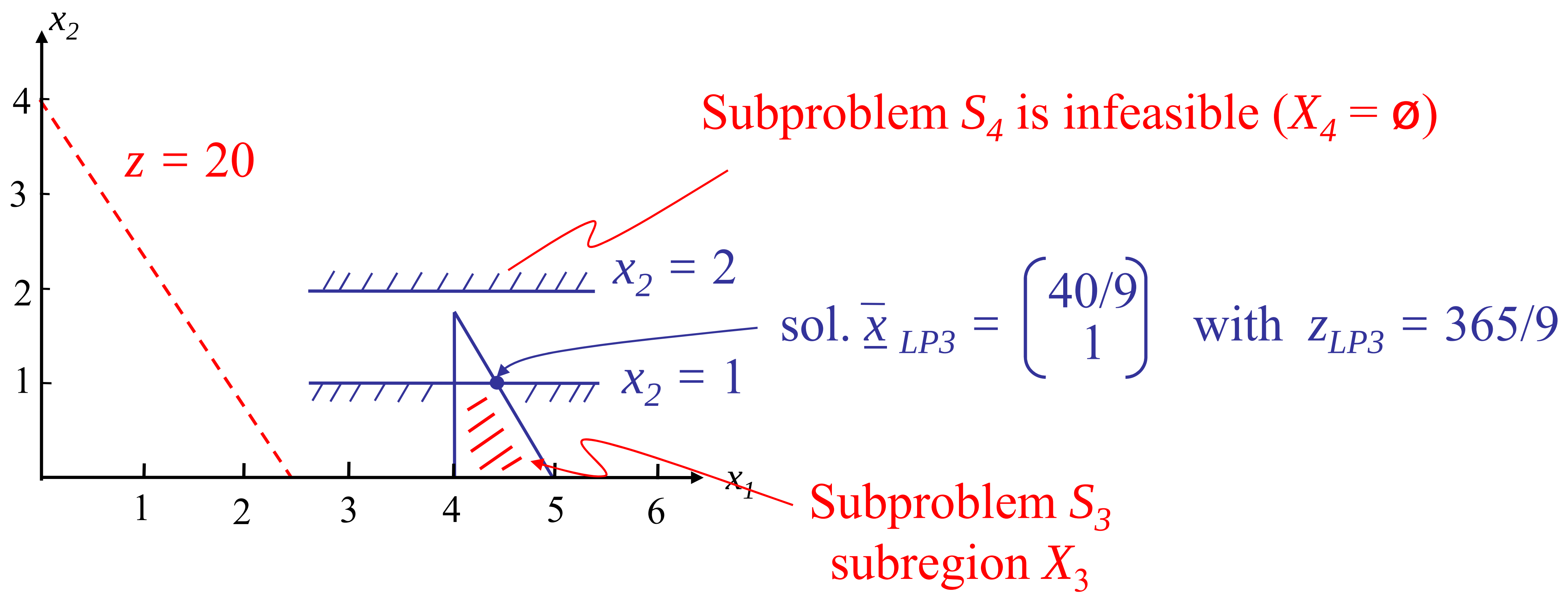
After considering X_1, best feasible (integer) solution found so far:

$$\underline{x}' = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \quad \text{with} \quad z' = 39.$$

Since $z_{LP2} = 41 > 39$, X_2 may contain a better feasible solution of ILP.

$\Rightarrow$ Partition X_2 into X_3 and X_4 by imposing:

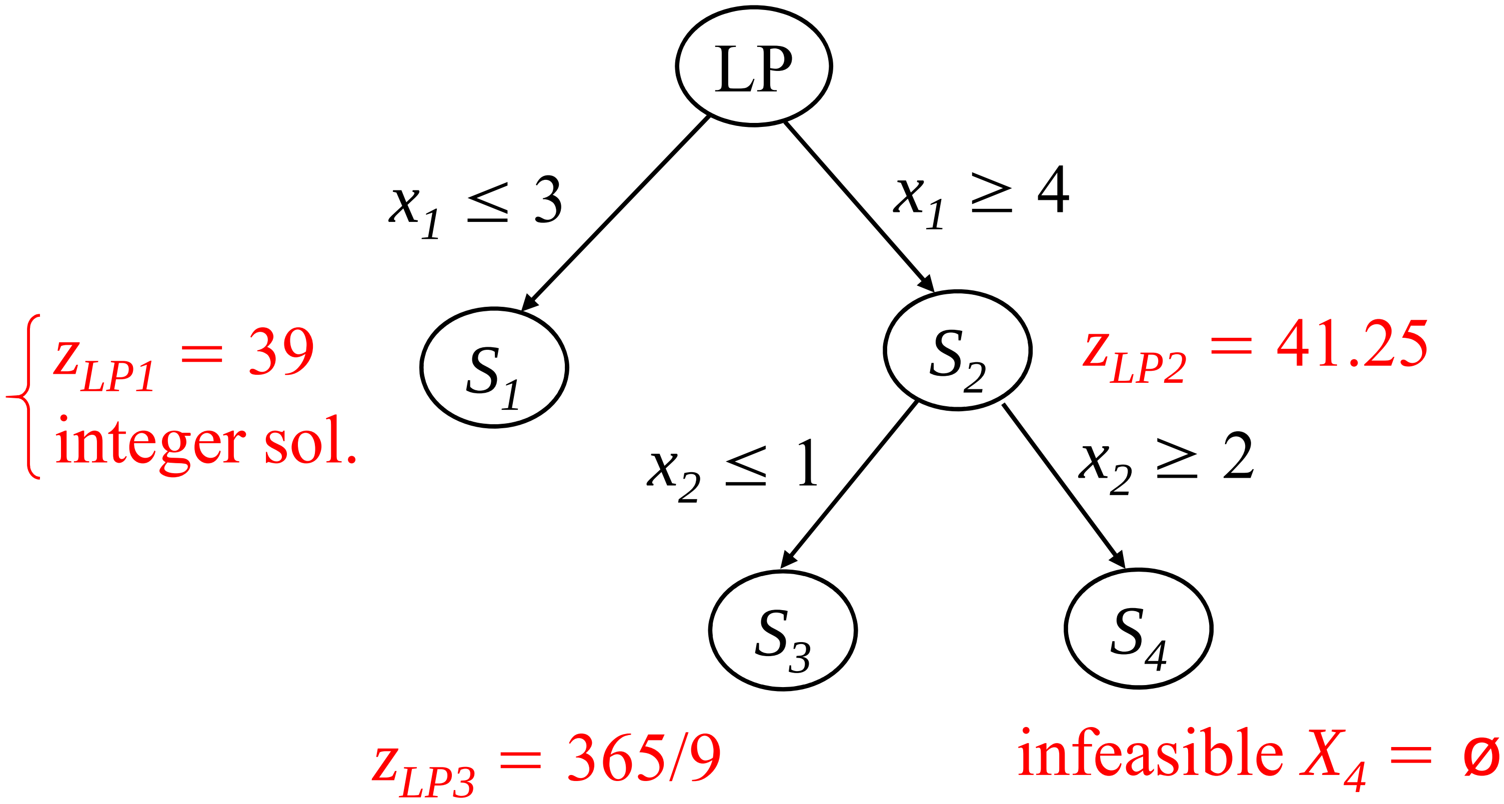
$$x_2 \leq \lfloor \bar{x}_2 \rfloor = 1 \qquad \text{or} \qquad x_2 \geq \lceil \bar{x}_2 \rceil + 1 = 2$$



x_2
x_1
4
3
2
1
1
2
3
4
5
6
$z = 20$
Subproblem S_4 is infeasible ($X_4 = \varnothing$)
$x_2 = 2$
$x_2 = 1$
sol. $\underline{\bar{x}}_{LP3} = \begin{pmatrix} 40/9 \\ 1 \end{pmatrix}$ with $z_{LP3} = 365/9$
Subproblem S_3
subregion X_3

Branching tree:

Best feasible solution found so far

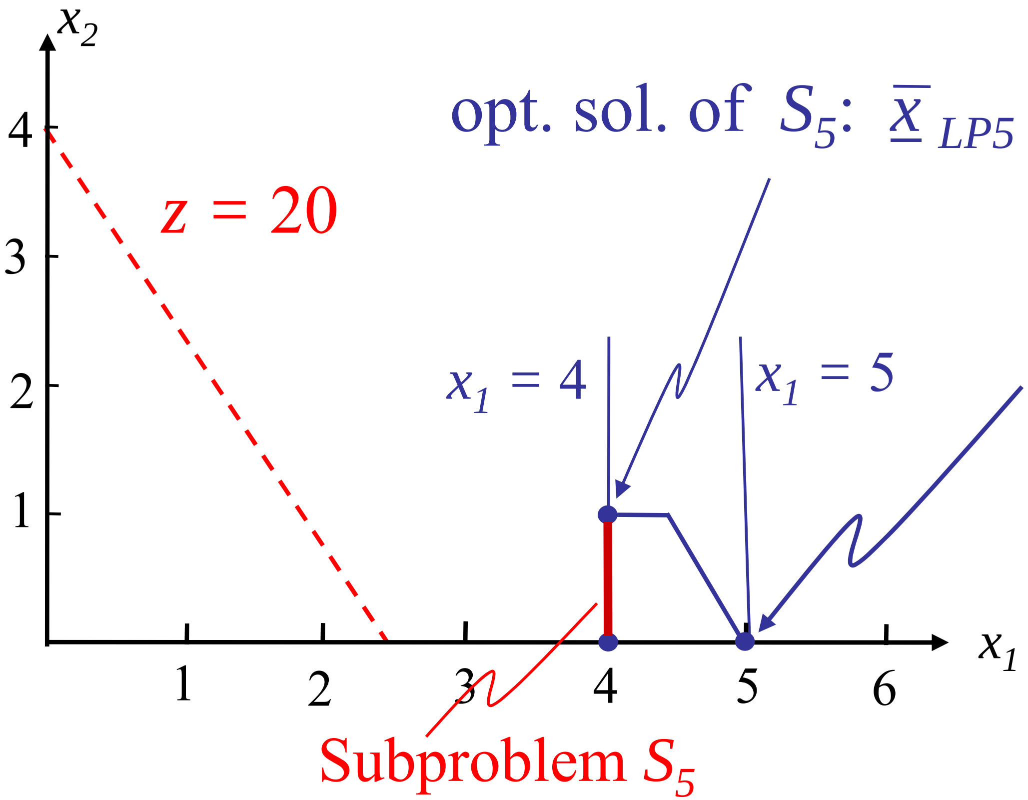


Since $z_{LP3} = 365/9 > 39$, X_3 may contain a better feasible solution of ILP.

$\Rightarrow$ Partition X_3 into X_5 and X_6 by imposing:

$$\underline{\overline{x}}_{LP3} = \begin{pmatrix} 40/9 \\ 1 \end{pmatrix} \qquad x_1 \leq \lfloor \overline{x}_1 \rfloor = 4 \quad \text{or} \quad x_1 \geq \lfloor \overline{x}_1 \rfloor + 1 = 5$$

opt. sol. of S_5: $\overline{\underline{x}}_{LP5} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ integer with $z_{LP5} = 37$



unique feasible solution of S_6

$$\begin{cases} \overline{\underline{x}}_{LP6} = \begin{bmatrix} 5 \\ 0 \end{bmatrix} & \text{integer} \\ z_{LP6} = \underline{40} & \end{cases}$$

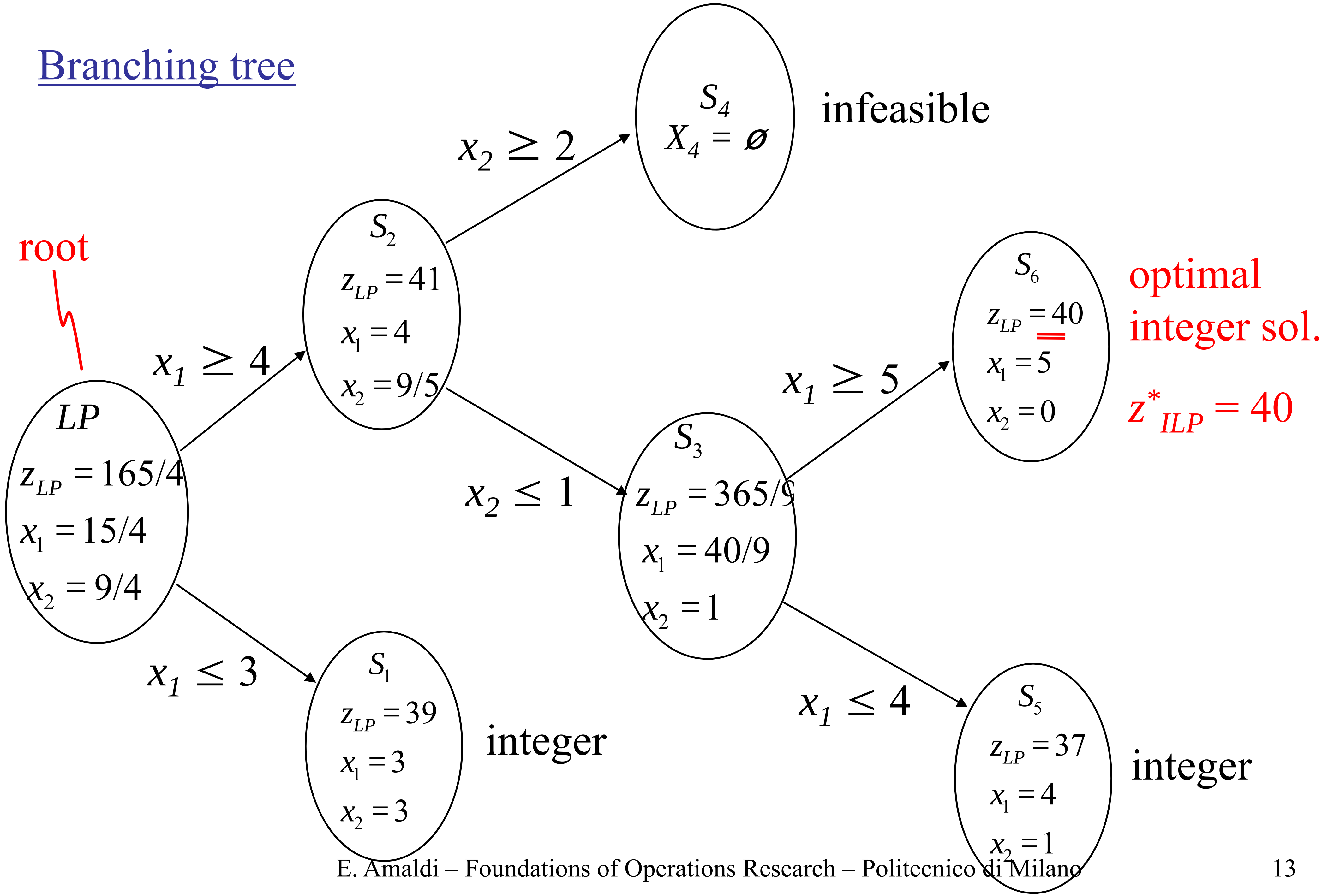
Integer solution $\overline{\underline{x}}_{LP5}$ (feasible for ILP) but with worse obj. fct. value of

$$\underline{x}' = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \quad \text{with } z' = 39.$$

$\overline{\underline{x}}_{LP6}$ is the best integer solution found $\Rightarrow$ <u>optimal solution.</u>

|| Branch & Bound is an exact method (it guarantees an optimal solution).

Branching tree



The branching tree may not contain all possible nodes

(2^d leaves)

A node of the tree has no child – is “fathomed”– if

- initial constraints + those on the arcs from the root are infeasible (e.g. S_4)
- optimal solution of the linear relaxation is integer (e.g. S_1)
- the value $\underline{c}^T\underline{\overline{x}}_{LP}$ of the optimal solution $\underline{\overline{x}}_{LP}$ of the linear relaxation is worse than that of the best feasible solution of ILP found so far.

≡ Bounding criterion

Observation: In the third case the feasible subregion of the subproblem associated to that node cannot contain an integer feasible solution that is better than the best feasible solution of ILP found so far!

Bounding criterion often allows to "discard" a number of nodes (subproblems).

Choice of the node (subproblem) to examine:

- First deeper nodes (depth-first search strategy)

  Simple recursive procedure, it is easy to reoptimize but it may be costly in case of wrong choice.

- First more promising nodes (best-bound first strategy)

  with the best value of linear relaxation

  Typically generates a smaller number of nodes but suproblems are less constrained ⇒ takes longer to find a first feasible solution and to improve it.

Choice of the (fractional) variable for branching

- It may not be the best choice to select the variable x_h whose fractional value is closer to 0,5 (hoping to obtain two subproblems that are more stringent and balanced).

- Strong branching: try to branch on some of candidate variables (fractional basic ones), evaluate the corresponding objective function values, and actually branch on the variable that yields the best improvement in the objective function.

Efficient solution of the linear relaxations

No need to solve the linear relaxations of the ILP subproblems from scratch with, for instance, the two-phase Simplex algorithm.

To efficiently find an optimal solution of the strengthened linear relaxation with a new constraint, we can exploit the optimal tableau of the previous linear relaxation and apply a single iteration of the **Dual simplex method** (variant of the Simplex method not covered here).

Applicability of Branch and Bound approach

Branch & Bound is also applicable to mixed ILPs:

when branching just consider the fractional variables that must be integer.

General method that can be adapted to tackle any discrete optimization problem and many nonlinear optimization problems.

e.g., scheduling, traveling salesman problem,…

We "just" need

- Technique to partition a set of feasible solutions into two or more subsets of feasible solutions (branch).
- Procedure to determine a bound on the cost of any solution in such a subset of feasible solutions (bound).

Observation: Branch-and-Bound can also be used as a heuristic by imposing an upper bound on the computing time or on the number of nodes that are examined.