## 5.1 Branch-and-Bound

Given the integer linear program

$$\begin{aligned} \max z = 3x_1 + 4x_2 & \\ 2x_1 + x_2 & \le 6 \\ 2x_1 + 3x_2 & \le 9 \\ x_1, x_2 & \ge 0 \text{ integer} \end{aligned}$$

solve it via the Branch-and-Bound method (solving graphically the continuous relaxation of each subproblem encountered in the enumeration tree). Branch on the fractional variable with fractional value closest to $\frac{1}{2}$. Among the set of active nodes, pick that with the most promising bound.

## 5.2 Branch-and-Bound for 0-1 knapsack

A bank has 14 million Euro, which can be invested into stocks of four companies (1, 2, 3, and 4). The table reports, for each company, the net revenue and the amount of money that must be invested into it.

| Company | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Revenue | 16 | 22 | 12 | 8 |
| Money | 5 | 7 | 4 | 3 |

Give an Integer Linear Programming formulation for the problem of choosing a set of companies so as to maximize the total revenue. Note that no partial investment can be done, i.e., for each company we can either invest into it or not. Solve the problem with a Branch-and-Bound algorithm. Show that the continuous relaxation of the original problem and the resulting subproblems can be solved to optimality with a simple greedy algorithm.

## 5.3 Cutting plane algorithm

Given the integer linear program

$$\begin{aligned} \min \quad x_1 - 2x_2 & \\ -4x_1 + 6x_2 & \le 9 \\ x_1 + x_2 & \le 4 \\ x_1, x_2 & \ge 0 \text{ integer} \end{aligned}$$

solve it via the cutting plane method with Gomory's fractional cutting planes.

SOLUTION

## 5.1 Branch-and-Bound

The enumeration tree is reported in Figure 1. The graphical solution of each subproblem is reported. The subproblems are solved in the following order: P1, P2, P3, P4, P5, P6, P7. Note that when the optimal value $\bar{z}$ of a subproblem is fractional, we can round the upper bound given by the subproblem to $\lfloor \bar{z} \rfloor$. For instance, in P1 we obtain the bound $\lfloor \frac{51}{4} \rfloor = 12$.

After solving P7, we observe that P6 yields an integer solution which is worse than that of P7, which is therefore discarded. We also observe that P2 yields an upper bound which is smaller than the value of the best feasible solution found so far (in P7). The node is therefore pruned. The optimal solution (found in P7) is $x^* = (0, 3)$ with objective function value $z^* = 12$.

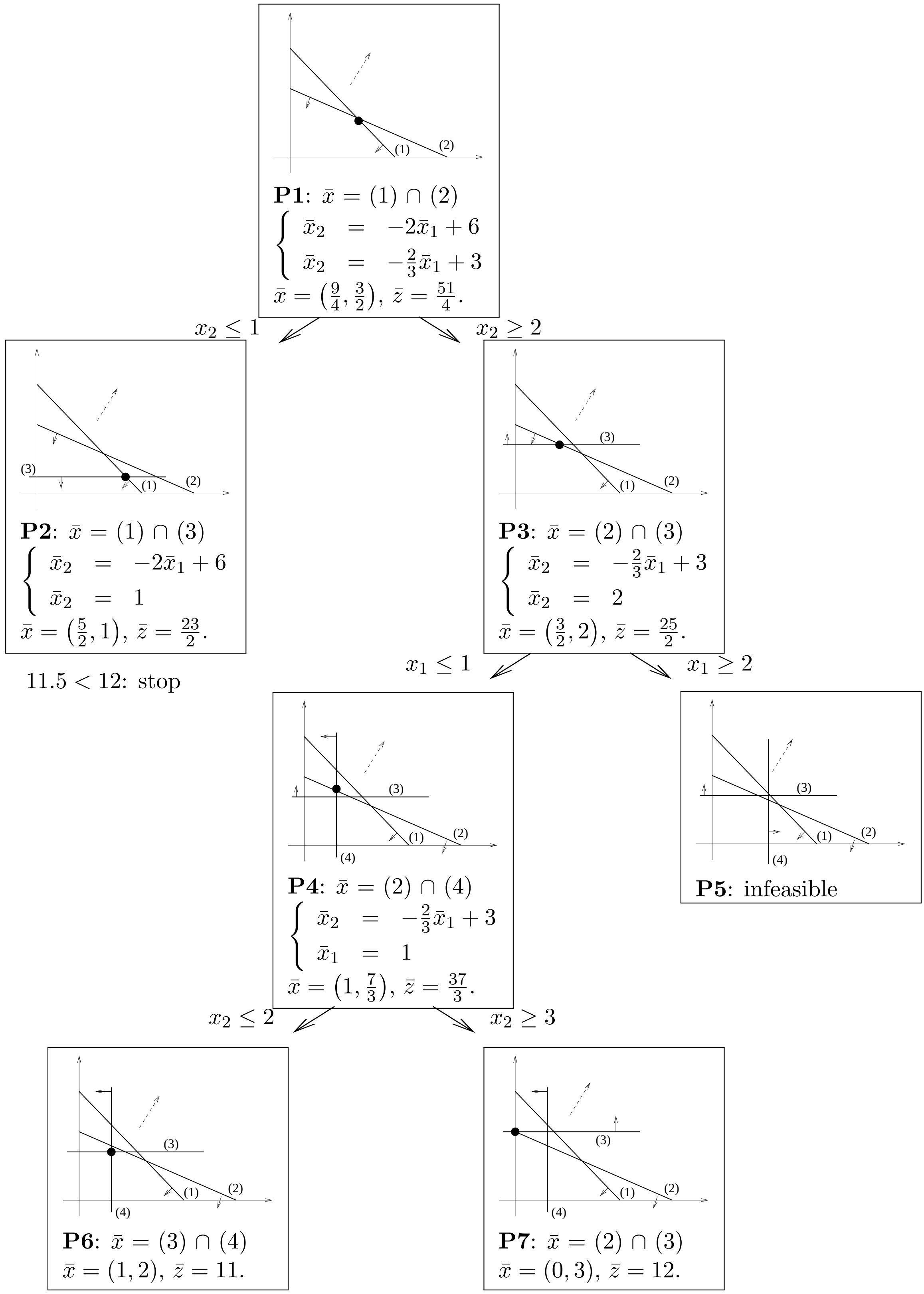


Figure 1: Enumeration tree for problem 5.1

## 5.2 Branch-and-Bound for 0-1 knapsack

The integer linear programming formulation for the problem is

$$\begin{aligned} \max 16x_1 + 22x_2 + 12x_3 + 8x_4 & \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 & \leq 14 \\ x_1, x_2, x_3, x_4 & \in \{0, 1\} \end{aligned}$$

and its linear programming relaxation is

$$\begin{aligned} \max 16x_1 + 22x_2 + 12x_3 + 8x_4 & \\ 5x_1 + 7x_2 + 4x_3 + 3x_4 & \leq 14 \\ 0 \leq x_i \leq 1 & \quad \forall i \in \{1, 2, 3, 4\}. \end{aligned}$$

To find an optimal solution to the linear programming relaxation of the knapsack problem, there is no need to use the two-phase Simplex method. We can use the following simple greedy algorithm. First, sort the variables by nonincreasing revenue and cost ratio. Note that in our case

$$(16/5, 22/7, 12/4, 8/3) = (3.2, 3.14, 3, 2.66)$$

and the variables are already ordered appropriately. Then, consider the variables in that order and assign the largest possible value to the variable under consideration $x_{i'}$ as long as $\sum_{i<i'} c_i \leq B$. More precisely, we set $x_{i'} = 1$ if $\sum_{i \leq i'} c_i \leq B$ and $x_{i'} = \frac{B - \sum_{i<i'} c_i}{c_{i'}}$ if $\sum_{i<i'} c_i + c_{i'} > B$, and all the other variables to zero.

For instance, at node 1 we have: $x_1 = 1$ (5 units are used), $x_2 = 1$ (7 units), $x_3 = \frac{1}{2}$ (2/4=1/2 units). Since, at each branching iteration, we set a variable either to 0 or 1, this greedy procedure for solving the linear programming relaxation of the knapsack problem can be applied in any node of the enumeration tree, by fixing the approriate variables.

The enumeration tree is given in Figure 2. Some observations:

- The index $t$ indicates the order by which the subproblems are solved.
- Since all variables are integer, whenever a subproblem yields a solution with fractional value, we round it to $\lfloor \bar{z} \rfloor$.
- The lower bounds (LB) is *not* computed at each node (to do this, a heuristic should be applied). We update it whenever a subproblem yields a feasible solution. Note that this value is NOT related to the specific subproblem, as it depends only on the iteration. Indeed, at each iteration $t$, LB corresponds to the value of the best feasible solution found so far in any part of the enumeration tree. For instance, in subproblem 4 we find a feasible solution of value $\bar{z} = 36$. Since it is the first that is found and LB still has the initial value of $-\infty$, we set LB to 36.
- In subproblem 6 an integer solution is found and the node is *pruned by feasibility.*
- Subproblem 7 is infeasible, since $\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = 1$ require a budget of $16 > 14$. The node is *pruned by infeasibility.*

- Subproblem 8 yields an upper bound of $\bar{z} = 38$ which is strictly smaller than the current LB of 42. The node is *pruned by bound.*
- The same happens for subproblem 9, where $\bar{z} = 42 + \frac{6}{7}$. The upper bound is $\lfloor \bar{z} \rfloor = 42$, which is strictly smaller than the current LB of value 42. Node 9 is *pruned by bound.*

The final optimal solution, which is found in node 6, is $x^* = (0, 1, 1, 1)$, of value 42.

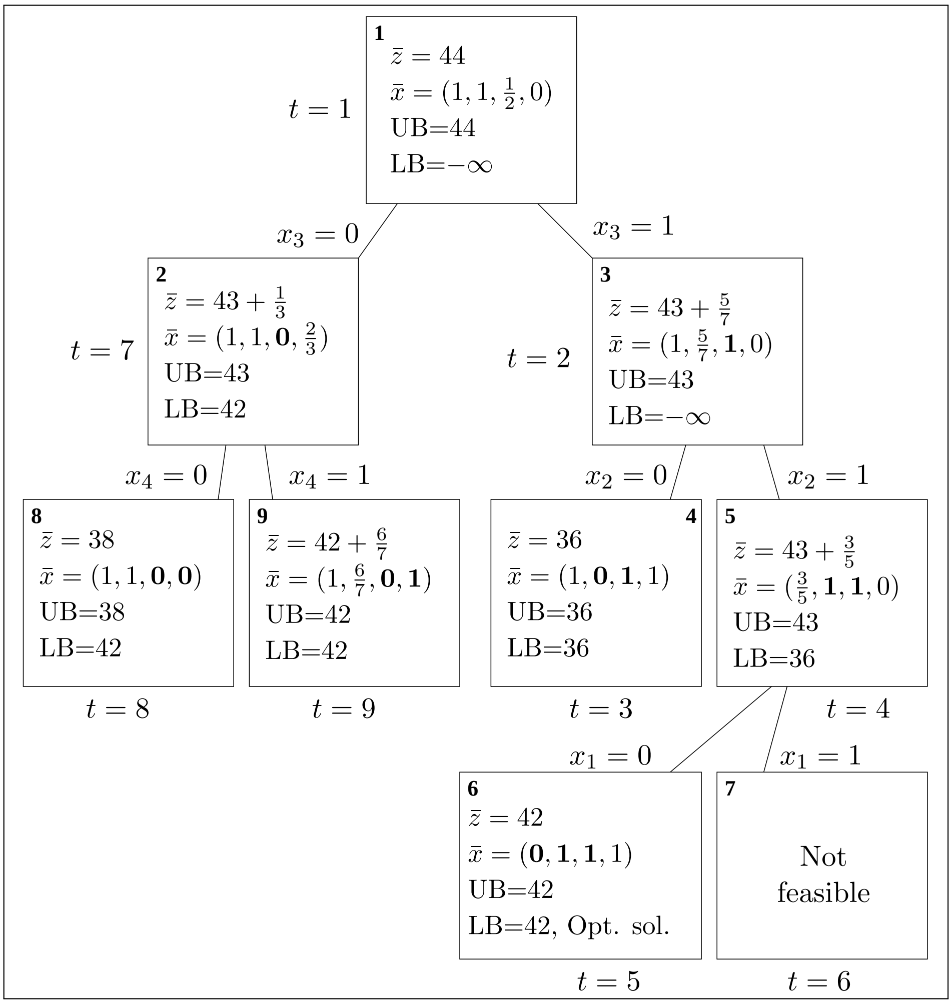


Figure 2: Enumeration tree for problem 5.2

## 5.3 Cutting plane algorithm

The continuous relaxation of the problem at hand, reduced to standard form, reads

$$\begin{aligned}\min \quad & x_1 - 2x_2 \\ & -4x_1 + 6x_2 + x_3 = 9 \\ & x_1 + x_2 + x_4 \quad = 4 \\ & x_1, x_2, x_3, x_4 \quad \geq 0,\end{aligned}$$

were $x_3, x_4$ are slack variables.

We solve it via the primal simplex method. The initial feasible basic solution is $x_B = (x_3, x_4)$. We obtain the following sequence of tableaux, where the pivot element is denoted by the symbol $\boxed{\cdot}$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| 0 | 1 | -2 | 0 | 0 |
| 9 | -4 | $\boxed{6}$ | 1 | 0 |
| 4 | 1 | 1 | 0 | 1 |

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| 3 | $-\frac{1}{3}$ | 0 | $\frac{1}{3}$ | 0 |
| $\frac{3}{2}$ | $-\frac{2}{3}$ | 1 | $\frac{1}{6}$ | 0 |
| $\frac{5}{2}$ | $\boxed{\frac{5}{3}}$ | 0 | $-\frac{1}{6}$ | 1 |

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| $\frac{7}{2}$ | 0 | 0 | $\frac{3}{10}$ | $\frac{1}{5}$ |
| $\frac{5}{2}$ | 0 | 1 | $\frac{1}{10}$ | $\frac{2}{5}$ |
| $\frac{3}{2}$ | 1 | 0 | $-\frac{1}{10}$ | $\frac{3}{5}$ |

The optimal solution to the linear programming relaxation is $\bar{x} = (\frac{3}{2}, \frac{5}{2})$, where $x_3 = x_4 = 0$ (see Figure 3).

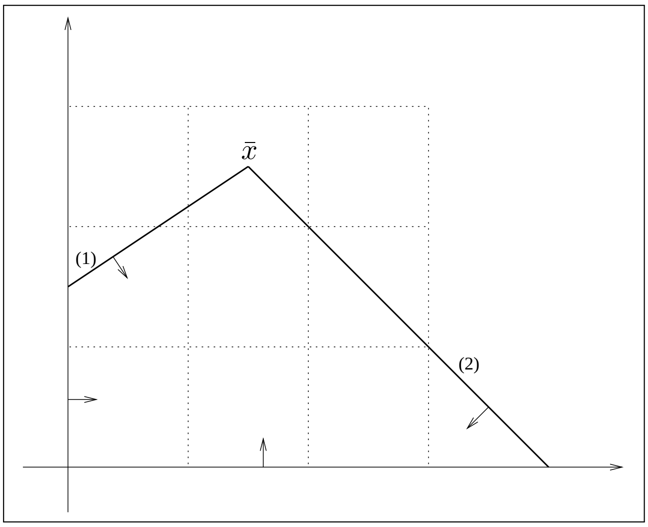


Figure 3: Graphical solution to problem 5.3

We derive a Gomory fractional cut from the first row of the optimal tableau $x_2 + \frac{1}{10}x_3 + \frac{2}{5}x_4 = \frac{5}{2}$.

The cut is defined as

$$x_i + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor, \tag{1}$$

where $F$ is the set of the indices of the nonbasic variables and $i$ is the index of the basic variable corresponding to the tableau row that is chosen. We obtain the cut $x_2 \leq 2$ (see Figure 4 (constraint (3)).

The cut is to be added to the tableau. Note that, in the current form, it is *not* a function of the nonbasic variables $x_3, x_4$. Instead of adding it to the tableau and performing some pivot operations to restore the correct form of the tableau, we can write the fractional form of the cut. Considering the $i$th row of the optimal tableau

$$x_i + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{b}_i$$

and subtracting from it the cut (1), we obtain

$$\sum_{j \in N} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j \geq (\bar{b}_i - \lfloor \bar{b}_i \rfloor).$$

In our case, we have

$$\frac{1}{10} x_3 + \frac{2}{5} x_4 \geq \frac{1}{2}$$

which, by introducing a surplus variable $x_5 \geq 0$ becomes

$$\frac{1}{10} x_3 + \frac{2}{5} x_4 - x_5 = \frac{1}{2}.$$

Observe that $x_5$ only occurs in the new row. Therefore, it is directly added to the set of basic variables. We multiply the cut by -1, obtaining

$$-\frac{1}{10} x_3 - \frac{2}{5} x_4 + x_5 = -\frac{1}{2}.$$

We obtain the new tableau

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|
| $\frac{7}{2}$ | 0 | 0 | $\frac{3}{10}$ | $\frac{1}{5}$ | 0 |
| $\frac{5}{2}$ | 0 | 1 | $\frac{1}{10}$ | $\frac{2}{5}$ | 0 |
| $\frac{3}{2}$ | 1 | 0 | $-\frac{1}{10}$ | $\frac{3}{5}$ | 0 |
| $-\frac{1}{2}$ | 0 | 0 | $-\frac{1}{10}$ | $-\frac{2}{5}$ | 1 |

We reoptimize the tableau via a single iteration of the dual simplex algorithm (involving an appropriate rule, not covered in this course, for selecting the non basic variable that enters the basis). We perform a pivoting operation on the highlighted element

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|
| $\frac{7}{2}$ | 0 | 0 | $\frac{3}{10}$ | $\frac{1}{5}$ | 0 |
| $\frac{5}{2}$ | 0 | 1 | $\frac{1}{10}$ | $\frac{2}{5}$ | 0 |
| $\frac{3}{2}$ | 1 | 0 | $-\frac{1}{10}$ | $\frac{3}{5}$ | 0 |
| $-\frac{1}{2}$ | 0 | 0 | $-\frac{1}{10}$ | $\boxed{-\frac{2}{5}}$ | 1 |

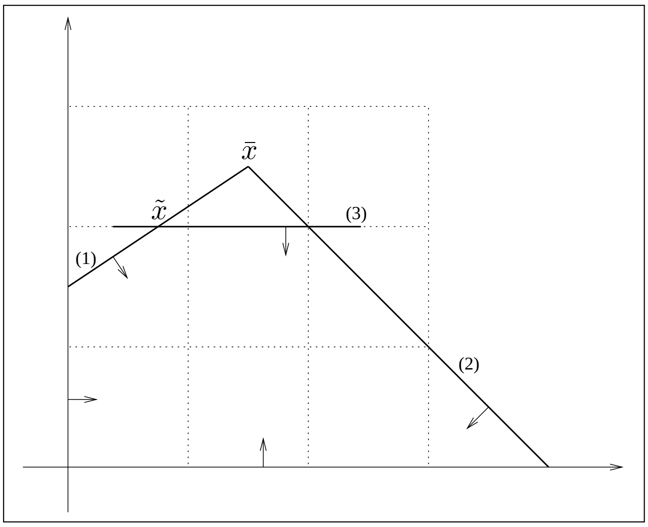


Figure 4: First Gomory cut for problem 5.3

obtaining the following optimal tableau of the linear programming relaxation

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|
| $\frac{13}{4}$ | 0 | 0 | $\frac{1}{4}$ | 0 | $\frac{1}{2}$ |
| 2 | 0 | 1 | 0 | 0 | 1 |
| $\frac{3}{4}$ | 1 | 0 | $-\frac{1}{4}$ | 0 | $\frac{3}{2}$ |
| $\frac{5}{4}$ | 0 | 0 | $\frac{1}{4}$ | 1 | $-\frac{5}{2}$ |

corresponding to the optimal LP solution $\tilde{x} = (\frac{3}{4}, 2)$. Since it is not integer, we perform another iteration of the cutting plane method.

We pick the second row

$$x_1 - \frac{1}{4}x_3 + \frac{3}{2}x_5 = \frac{3}{4},$$

from which we deduce the Gomory fractional cut $x_1 - x_3 + x_5 \leq 0$ which, in the space of the original variable, amounts to $-3x_1 + 5x_2 \leq 7$. Its fractional version is

$$\frac{3}{4}x_3 + \frac{1}{2}x_5 \geq \frac{3}{4}.$$

We obtain the tableau

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| $\frac{13}{4}$ | 0 | 0 | $\frac{1}{4}$ | 0 | $\frac{1}{2}$ | 0 |
| 2 | 0 | 1 | 0 | 0 | 1 | 0 |
| $\frac{3}{4}$ | 1 | 0 | $-\frac{1}{4}$ | 0 | $\frac{3}{2}$ | 0 |
| $\frac{5}{4}$ | 0 | 0 | $\frac{1}{4}$ | 1 | $-\frac{5}{2}$ | 0 |
| $-\frac{3}{4}$ | 0 | 0 | $-\frac{3}{4}$ | 0 | $-\frac{1}{2}$ | 1 |

Performing the pivot operation

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| $\frac{13}{4}$ | 0 | 0 | $\frac{1}{4}$ | 0 | $\frac{1}{2}$ | 0 |
| 2 | 0 | 1 | 0 | 0 | 1 | 0 |
| $\frac{3}{4}$ | 1 | 0 | $-\frac{1}{4}$ | 0 | $\frac{3}{2}$ | 0 |
| $\frac{5}{4}$ | 0 | 0 | $\frac{1}{4}$ | 1 | $-\frac{5}{2}$ | 0 |
| $-\frac{3}{4}$ | 0 | 0 | $\boxed{-\frac{3}{4}}$ | 0 | $-\frac{1}{2}$ | 1 |

we obtain the tableau

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|---|
| 3 | 0 | 0 | 0 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ |
| 2 | 0 | 1 | 0 | 0 | 1 | 0 |
| 1 | 1 | 0 | 0 | 0 | $\frac{5}{3}$ | $-\frac{1}{3}$ |
| 1 | 0 | 0 | 0 | 1 | $-\frac{8}{3}$ | $\frac{1}{3}$ |
| 1 | 0 | 0 | 1 | 0 | $\frac{2}{3}$ | $-\frac{4}{3}$ |

which yields the integer solution $x^* = (1, 2)$, shown in Figure 5 (together with the last Gomory fractional cut that was added).

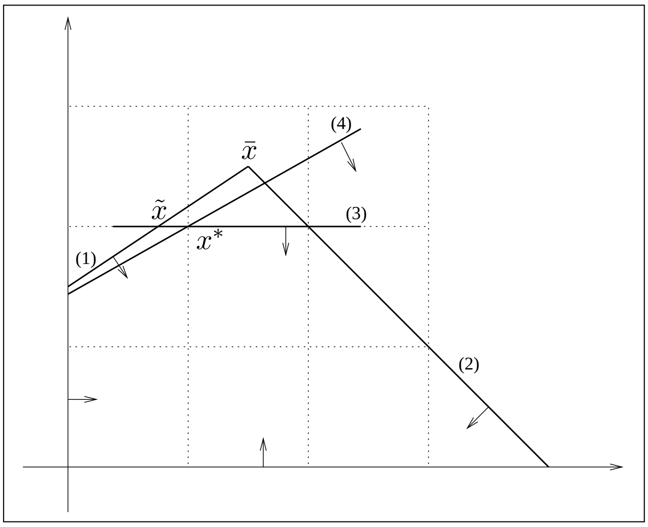


Figure 5: Last Gomory cut for problem 5.3