4.4 Linear programming dual

Write the dual of the following linear program

$$
\begin{array}{lrrrrrrrrl}
\min & & & 2x_2 & + & x_3 & - & 3x_4 & & \\
& x_1 & - & x_2 & & & + & 2x_4 & \geq & 2 \\
& & & 2x_2 & + & x_3 & & & = & 4 \\
& 2x_1 & & & - & x_3 & + & x_4 & \leq & 1 \\
& & & & & & & x_1 & \geq & 0 \\
& & & & & & & x_2 & \geq & 0 \\
& & & & & & x_3, & x_4 & & \text{unrestricted}
\end{array}
$$

4.5 Dual of the transportation problem and its economic interpretation

A company produces a single type of good in m factories and stores it in n warehouses. Suppose that the factory i, with $i \in \{1, \ldots, m\}$, has a production capacity of b_i units, and the warehouse j, with $j \in \{1, \ldots, n\}$, requires d_j units. Let c_{ij} be the unit transportation cost from factory i to warehouse j.

Consider the problem of determining a transportation plan that minimizes the (transportation) costs, while satisfying the factory capacities and the warehouse demands.

If x_{ij} denotes the amount of product transported from factory i to warehouse j, we have the following Linear Programming formulation:

$$
\begin{array}{lrll}
\min & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} & & \\
& -\sum_{j=1}^{n} x_{ij} & \geq -b_i & i \in \{1, \ldots, m\} \\
& \sum_{i=1}^{m} x_{ij} & \geq d_j & j \in \{1, \ldots, n\} \\
& x_{ij} & \geq 0 & i \in \{1, \ldots, m\} \quad j \in \{1, \ldots, n\}.
\end{array}
$$

Without loss of generality, we assume that the total production capacity satisfies $\sum_{i=1}^{m} b_i = \sum_{j=1}^{n} d_j$. Indeed, if $\sum_{i=1}^{m} b_i > \sum_{j=1}^{n} d_j$, we can introduce a dummy $(n+1)$-th warehouse with demand $\sum_{i=1}^{m} b_i - \sum_{j=1}^{n} d_j$ and zero transportation costs.

Determine the dual of the above Linear Programming formulation and provide an economic interpretation of the dual.

4.6 Complementary slackness conditions

Given the linear programm

$$
\begin{array}{lrcrcr}
\text{(P)} \quad \max & 2x_1 & + & x_2 & & \\
 & x_1 & + & 2x_2 & \leq & 14 \\
 & 2x_1 & - & x_2 & \leq & 10 \\
 & x_1 & - & x_2 & \leq & 3 \\
 & & x_1, & x_2 & \geq & 0
\end{array}
$$

a) write its dual,

b) check that $\underline{\bar{x}} = (\frac{20}{3}, \frac{11}{3})$ is a feasible solution of (P),

c) show that $\underline{\bar{x}}$ is also an optimal solution (P) by applying the complementary slackness conditions,

d) derive an optimal solution of the dual.

SOLUTION

4.4 **Linear programming dual**

The dual of the given linear program is

$$
\begin{array}{lrcrcrcl}
\max & 2y_1 & + & 4y_2 & + & y_3 & & \\
 & y_1 & & & + & 2y_3 & \leq & 0 \\
 & -y_1 & + & 2y_2 & & & \leq & 2 \\
 & & & y_2 & - & y_3 & = & 1 \\
 & 2y_1 & & & + & y_3 & = & -3 \\
 & & & & & y_1 & \geq & 0 \\
 & & & & & y_2 & & \text{unrestricted} \\
 & & & & & y_3 & \leq & 0.
\end{array}
$$

4.5 **Dual of the transportation problem and its economic interpretation**

Let u_i and v_j, with $i \in I$ and with $j \in J$, be the dual variables associated to the two classes of constraints. Since in each column of the constraint matrix of the primal problem there are exactly two non-zero entries, with value -1 and +1, appearing in rows corresponding to the fist and, respectively, the second class of constraints, we have the following dual problem:

$$
\begin{array}{lll}
(\mathbf{D_1}) \quad \max -\sum_{i=1}^{m} b_i u_i + \sum_{j=1}^{n} d_j v_j & & \\
\qquad v_j - u_i \leq c_{ij} & i \in \{1, \ldots, m\} & j \in \{1, \ldots, n\} \\
\qquad u_i \geq 0, v_j \geq 0 & i \in \{1, \ldots, m\} & j \in \{1, \ldots, n\}
\end{array}
$$

Economic interpretation. We suppose that the production company (company A) hires a logistics company to handle the transportation business (company B). This company buys all the products from the different factories, paying a unit price (cost) of u_i Euro at factory i. Then, it sells the products to the warehouses at a unit price of v_j Euro for warehouse j. The objective of company B is clearly to maximize the profit, that is

$$\max -\sum_{i=1}^{m} b_i u_i + \sum_{j=1}^{n} d_j v_j.$$

Company B has to decide the prices u_i and v_j. These prices must clearly be competitive, that is, $v_j - u_i \leq c_{ij}$ for each pair i, j. Indeed, if for any pair i, j the prices are such that $v_j - u_i > c_{ij}$, the company A will not hire company B and will take care of the transportation directly.

4.6 **Complementary slackness conditions**

a) The dual of the given linear program is

$$(D_2) \quad \begin{array}{rrrrrrl} \min & 14y_1 & + & 10y_2 & + & 3y_3 & \\ & y_1 & + & 2y_2 & + & y_3 & \geq 2 \\ & 2y_1 & - & y_2 & - & y_3 & \geq 1 \\ & & & y_1, y_2, y_3 & \geq 0 & & \end{array}$$

b) $\underline{\bar{x}} = (\frac{20}{3}, \frac{11}{3})$ is feasible, as it satisfies the constrains of (P_2).

c-d) According to the complementary slackness theorem, a feasible solution $\underline{\bar{x}} = (x_1, x_2)$ for the primal and a feasible solution $\underline{\bar{y}} = (y_1, y_2, y_3)$ for the dual are optimal for the respective problems if and only if they satisfy

$$y_i(a_i^T \underline{x} - b_i) = 0 \qquad \forall i$$

$$(c_j - \underline{y}^T A_j)x_j = 0 \qquad \forall j.$$

The complementary slackness conditions for the problem at hand are

$$y_1(x_1 + 2x_2 - 14) = 0 \tag{1}$$
$$y_2(2x_1 - x_2 - 10) = 0 \tag{2}$$
$$y_3(x_1 - x_2 - 3) = 0 \tag{3}$$
$$x_1(y_1 + 2y_2 + y_3 - 2) = 0 \tag{4}$$
$$x_2(2y_1 - y_2 - y_3 - 1) = 0. \tag{5}$$

We obtain $\underline{\bar{y}}$ by substituting for $\underline{\bar{x}}$. Since $\underline{\bar{x}} = (\frac{20}{3}, \frac{11}{3})$ satisfies as equations the first and third constraints of the primal but not the second constraint, we deduce $y_2 = 0$. Since $x_1 > 0$ and $x_2 > 0$, we obtain

$$y_1 + 2y_2 + y_3 - 2 = 0 \tag{6}$$
$$2y_1 - y_2 - y_3 - 1 = 0 \tag{7}$$
$$y_2 = 0 \tag{8}$$

By solving the system, we obtain $\underline{\bar{y}} = (1, 0, 1)$, which satisfies the dual constraints of D_2. Since $\underline{\bar{x}}$ is primal feasible and $\underline{\bar{y}}$ is dual feasible, the primal/dual pair $(\underline{\bar{x}}, \underline{\bar{y}})$ satisfies the complementary slackness constraints and, therefore, $\bar{x}$ is an optimal solution of the primal and $\underline{\bar{y}}$ is an optimal solution of the dual.

To double check, note that the objective function values of the respective problems of the two solutions are equal, namely, we have $\underline{c}^T \underline{x} = \underline{y}^T \underline{b}$.