4.4 Linear programming dual

Write the dual of the following linear program

$$
\begin{array}{lrcrcrcrcl}
\min & & & 2x_2 & + & x_3 & - & 3x_4 & & \\
& x_1 & - & x_2 & & & + & 2x_4 & \geq & 2 \\
& & & 2x_2 & + & x_3 & & & = & 4 \\
& 2x_1 & & & - & x_3 & + & x_4 & \leq & 1 \\
& & & & & & & x_1 & \geq & 0 \\
& & & & & & & x_2 & \geq & 0 \\
& & & & & & & x_3, & x_4 & \text{unrestricted}
\end{array}
$$

4.5 Dual of the transportation problem and its economic interpretation

A company produces a single type of good in m factories and stores it in n warehouses. Suppose that the factory i, with $i \in \{1, \dots, m\}$, has a production capacity of b_i units, and the warehouse j, with $j \in \{1, \dots, n\}$, requires d_j units. Let c_{ij} be the unit transportation cost from factory i to warehouse j.

Consider the problem of determining a transportation plan that minimizes the (transportation) costs, while satisfying the factory capacities and the warehouse demands.

If x_{ij} denotes the quantity of product transported from factory i to warehouse j, we have the following Linear Programming formulation:

$$
\begin{array}{llll}
\min & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} & & \\
& -\sum_{j=1}^{n} x_{ij} \geq -b_i & i \in \{1, \dots, m\} & \\
& \sum_{i=1}^{m} x_{ij} \geq d_j & j \in \{1, \dots, n\} & \\
& x_{ij} \geq 0 & i \in \{1, \dots, m\} & j \in \{1, \dots, n\}.
\end{array}
$$

Without loss of generality, we assume that the total production capacity satisfies $\sum_{i=1}^{m} b_i = \sum_{j=1}^{n} d_j$. Indeed, if $\sum_{i=1}^{m} b_i > \sum_{j=1}^{n} d_j$, we can introduce a dummy $(n+1)$-th warehouse with demand $\sum_{i=1}^{m} b_i - \sum_{j=1}^{n} d_j$ and zero transportation costs.

Determine the dual of the above Linear Programming formulation and provide an economic interpretation of the dual.

4.6 Complementary slackness conditions

Given the linear programm

$$
\begin{array}{llrcrcr}
(\mathrm{P}) & \max & 2x_1 & + & x_2 & & \\
 & & x_1 & + & 2x_2 & \leq & 14 \\
 & & 2x_1 & - & x_2 & \leq & 10 \\
 & & x_1 & - & x_2 & \leq & 3 \\
 & & & x_1, & x_2 & \geq & 0
\end{array}
$$

a) write its dual,

b) check that $\underline{\bar{x}} = (\frac{20}{3}, \frac{11}{3})$ is a feasible solution of (P),

c) show that $\underline{\bar{x}}$ is also an optimal solution (P) by applying the complementary slackness conditions,

d) derive an optimal solution of the dual.