4.1 Graphical solution of a linear program and standard form

Consider the problem

$$\begin{aligned} \min \quad & \underline{c}^T \underline{x} \\ & A\underline{x} \geq \underline{b} \\ & \underline{x} \geq \underline{0} \end{aligned}$$

where

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \underline{c} = \begin{pmatrix} 16 \\ 25 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 4 \\ 5 \\ 9 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 7 \\ 1 & 5 \\ 2 & 3 \end{pmatrix}$$

1. Solve the problem graphically and determine the basic and nonbasic variables of the optimal solution.
2. Put the problem in standard form w.r.t. the optimal basis (identify B, N, and the corresponding partition of the cost vector).

4.2 Geometry of linear programming

Consider the linear program

$$\begin{aligned} \max \quad z = \; & 3x_1 + 2x_2 \\ & 2x_1 + x_2 \leq 4 && (1) \\ & -2x_1 + x_2 \leq 2 && (2) \\ & x_1 - x_2 \leq 1 && (3) \\ & x_1, x_2 \geq 0 \end{aligned}$$

1. Solve it graphically and indicate the optimal solution and the corresponding objective function value.
2. Determine the basic feasible solutions (specifying the basic and nonbasic variables) corresponding to all vertices of the feasible region (polyhedron).
3. Indicate the sequence of basic feasible solutions that are visited by the simplex algorithm (let x_1 be the first variable to enter the basis) when starting from the initial basic feasible solution in which all slack variables are basic variables.
4. Determine the reduced costs for the basic feasible solutions associated to the vertices ((eq. 1) $\cap$ (eq. 2)) and ((eq. 1) $\cap$ (eq. 3)), where (eq. i) is obtained from (i), by substituting $\leq$ with $=$.
5. Show, geometrically, that the gradient of the objective function is a conic combination (i.e. a linear combination with nonnegative coefficients) of the gradients of the constraints which are active at an optimal vertex. Indicate the value taken by the objective function in that vertex. *Note*: All the constrains must be in $\leq$ form, since the problem is a maximization one (e.g., $x_1 \geq 0$ must be rewritten as $-x_1 \leq 0$).

6. Determine the range of values for the right hand side b_1 of constraint (1) for which the optimality of the basis solution is preserved.

7. Indicate for which values of the objective function coefficients $((x_1 = 0) \cap$ (eq. 2)) is an optimal vertex.

8. Determine the range of values for the right hand side b_2 of constraint (2) for which the feasible region is (a) empty, (b) contains a single point.

9. Indicate the range of values for c_1 for which there are multiple optimal solutions.

4.3 Simplex algorithm with Bland's rule

Solve the following linear program

$$
\begin{aligned}
\min \quad z = \quad & x_1 - 2x_2 && (4)\\
& 2x_1 \quad\quad + 3x_3 = 1 && (5)\\
& 3x_1 + 2x_2 - x_3 = 5 && (6)\\
& x_1, x_2, x_3 \geq 0 && (7)
\end{aligned}
$$

by applying the two-phase Simplex algorithm with Bland's rule.

SOLUTION

4.1 Graphical resolution and standard form

1. The equations associated to the system $A\underline{x} \geq \underline{b}$ are

$$x_1 + 7x_2 = 4 \tag{8}$$
$$x_1 + 5x_2 = 5 \tag{9}$$
$$2x_1 + 3x_2 = 9 \tag{10}$$

The corresponding lines are shown in Figure 1. The level curves (lines) of the objective

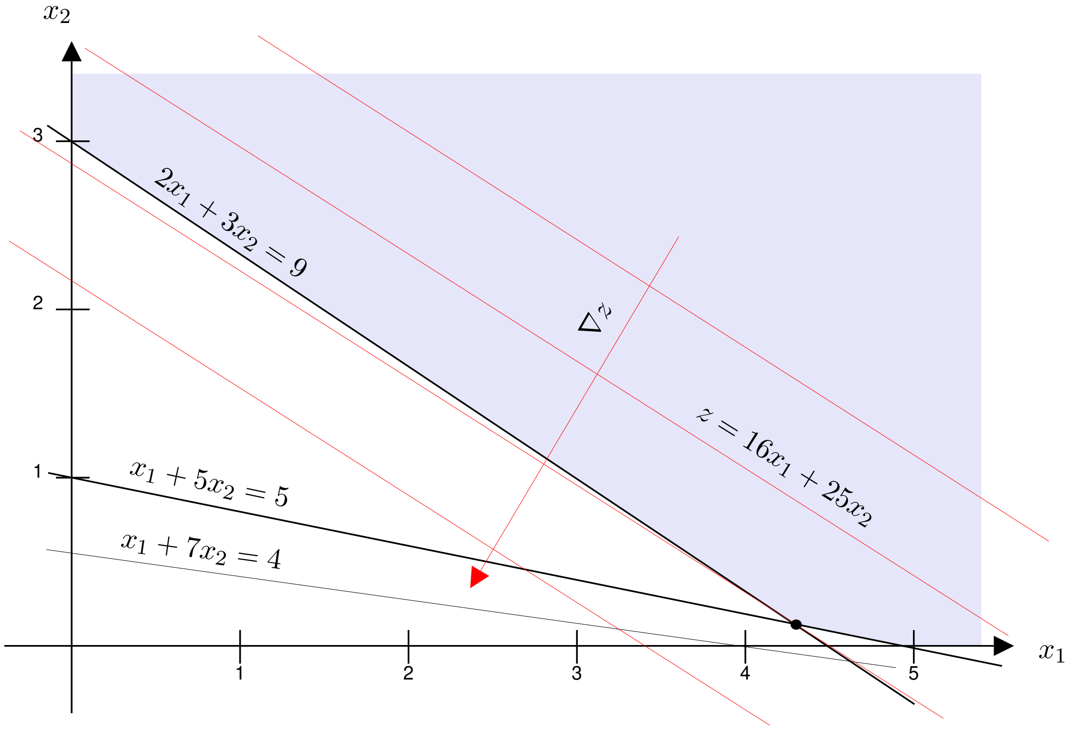

Figure 1: The polyhedron is unbounded

function are $z = 16x_1 + 25x_2$, or, equivalently, $x_2 = -\frac{16}{25}x_1 + \frac{z}{25}$. Inspecting Figure 2, we observe that vertex R is the unique optimal solution. Since vertex R is the intersection of equations (9) and (10), we obtain it by solving

$$x_1 + 5x_2 = 5$$
$$2x_1 + 3x_2 = 9$$

obtaining $x_1 = \frac{30}{7}, x_2 = \frac{1}{7}$.

2. To express the problem in standard form, we introduce 3 surplus variables, s_1, s_2, s_3, one

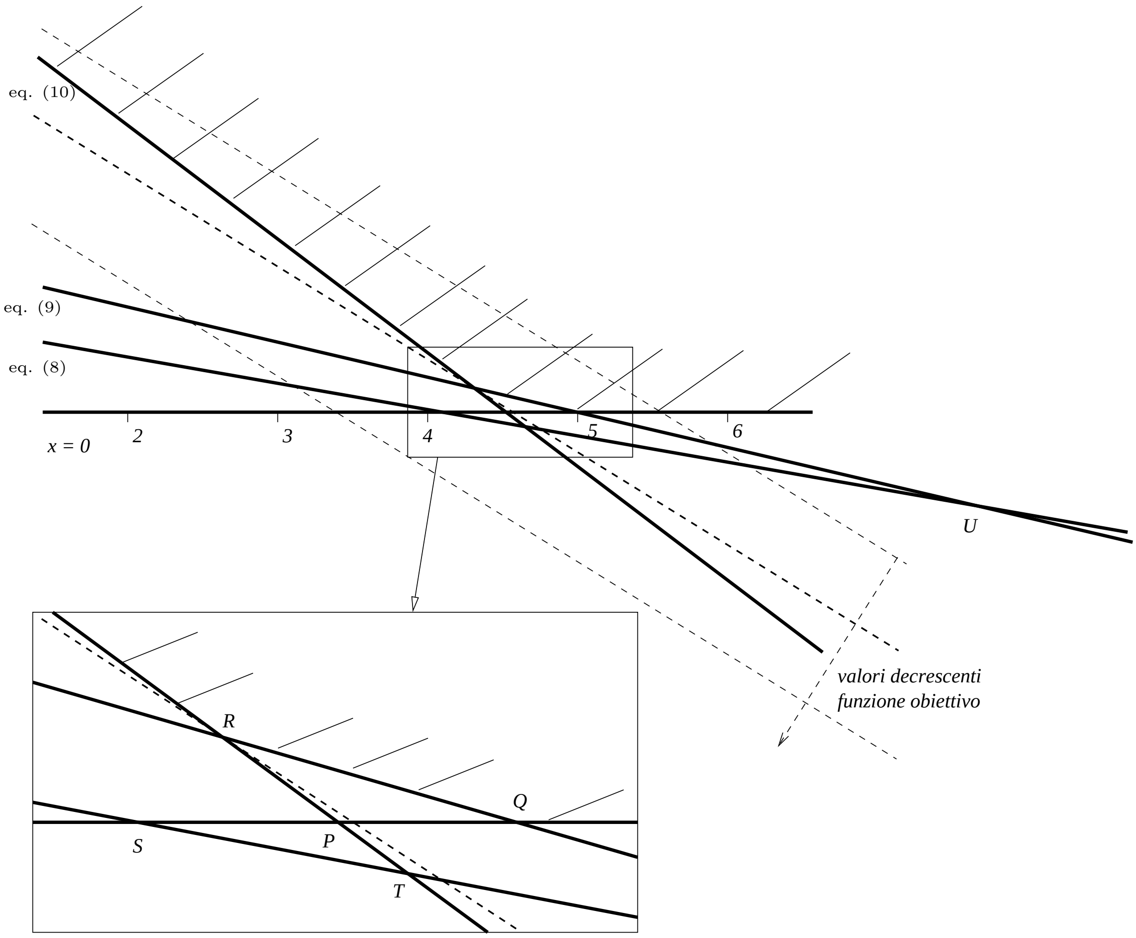

Figure 2: Graphical resolution: R is an optimal solution

per constraint. The new problem is

$$\begin{aligned} \min \quad & \underline{c}'^T \underline{x}' \\ A'\underline{x}' &= \underline{b} \\ \underline{x}' &\geq \underline{0} \end{aligned}$$

where $\underline{x}' = (x_1, x_2, s_1, s_2, s_3)^T$, $\underline{c}' = (16, 25, 0, 0, 0)$, and

$$A' = \left(\begin{array}{cc|ccc} 1 & 7 & -1 & 0 & 0 \\ 1 & 5 & 0 & -1 & 0 \\ 2 & 3 & 0 & 0 & -1 \end{array}\right) = (A| - I).$$

Since R is obtained by intersecting (9) and (10), the surplus variables of the corresponding constraints are equal to zero at R. By letting $\underline{x}_B = (x_1, x_2, s_1)$ and $\underline{x}_N = (s_2, s_3)$, we partition the variables into $\underline{x}' = (\underline{x}_B|\underline{x}_N)$. The corresponding partition of matrix A' is

$$A' = \left(\begin{array}{ccc|cc} 1 & 7 & -1 & 0 & 0 \\ 1 & 5 & 0 & -1 & 0 \\ 2 & 3 & 0 & 0 & -1 \end{array}\right) = (B|N).$$

The basic variables in R have values $\underline{x}_B = B^{-1}\underline{b}$. Since

$$B^{-1} = \frac{1}{7}\begin{pmatrix} 0 & -3 & 5 \\ 0 & 2 & -1 \\ -7 & 11 & -2 \end{pmatrix},$$

we have $\underline{x}_B = (x_1, x_2, s_1) = (\frac{30}{7}, \frac{1}{7}, \frac{9}{7})$.

4.2 Geometry of linear programming

1. The equations associated to constraints (1), (2), (3) are

$$\begin{aligned} 2x_1 + x_2 &= 4 \qquad \text{(eq. 1)} \\ -2x_1 + x_2 &= 2 \qquad \text{(eq. 2)} \\ x_1 - x_2 &= 1 \qquad \text{(eq. 3)} \end{aligned}$$

or, equivalently,

$$\begin{aligned} x_2 &= -2x_1 + 4 \\ x_2 &= 2x_1 + 2 \\ x_2 &= x_1 - 1. \end{aligned}$$

The level curves of the objective function are $z = 3x_1 + 2x_2$, i.e., $x_2 = -\frac{3x_1}{2} + \frac{z}{2}$. The polyhedron $PQROS$ of the feasible solutions is shown in Figure 3. The unique optimal solution is achieved at vertex $P = (\frac{1}{2}, 3)$, where $z^* = \frac{15}{2}$.

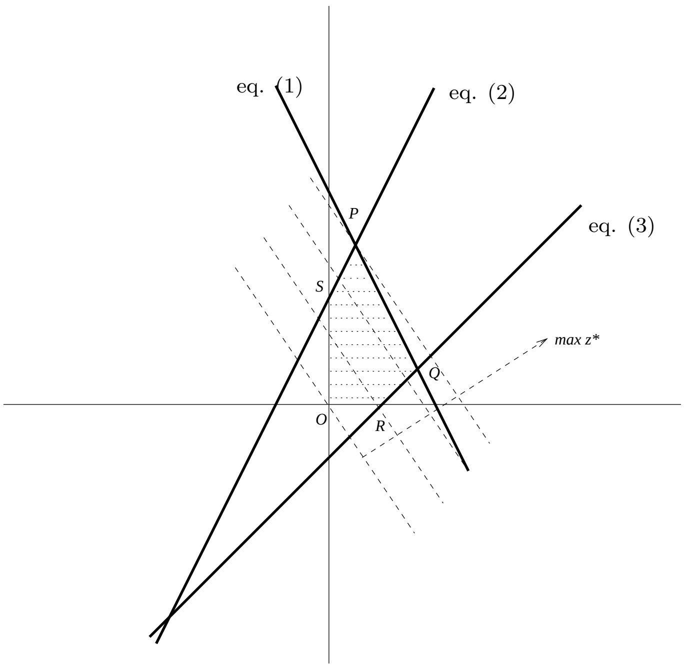

Figure 3: Polyhedron of the feasible solutions

2. To express the problem in standard form, we introduce 3 slack variables, s_1, s_2, s_3, one per constraint. The problem is

$$\begin{aligned} \max \quad & z = \underline{c}'^T \underline{x}' \\ & A'\underline{x}' = \underline{b} \\ & \underline{x}' \geq \underline{0}, \end{aligned}$$

where

$$\underline{x}' = \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}, \quad \underline{c}' = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{b}' = b, \quad A' = (A|I) = \left(\begin{array}{rr|rrr} 2 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{array}\right).$$

(a) Vertex P: $s_1 = 0, s_2 = 0$, where $\underline{x}_B = (x_1, x_2, s_3)$, $\underline{x}_N = (s_1, s_2)$,

$$B = \begin{pmatrix} 2 & 1 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

(b) Vertex Q: $s_1 = 0, s_3 = 0$, where $\underline{x}_B = (x_1, x_2, s_2)$, $\underline{x}_N = (s_1, s_3)$,

$$B = \begin{pmatrix} 2 & 1 & 0 \\ -2 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) Vertex R: $x_2 = 0, s_3 = 0$, where $\underline{x}_B = (x_1, s_1, s_2)$, $\underline{x}_N = (x_2, s_3)$,

$$B = \begin{pmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

(d) Vertex O: $x_1 = 0, x_2 = 0$, where $\underline{x}_B = (s_1, s_2, s_3)$, $\underline{x}_N = (x_1, x_2)$,

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 2 & 1 \\ -2 & 1 \\ 1 & -1 \end{pmatrix}.$$

(e) Vertex S: $x_1 = 0, s_2 = 0$, where $\underline{x}_B = (x_2, s_1, s_3)$, $\underline{x}_N = (x_1, s_2)$,

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 2 & 0 \\ -2 & 1 \\ 1 & 0 \end{pmatrix}.$$

3. Assume that, initially, the basic solution is given by the slack variables, i.e., $\underline{x}_B = (s_1, s_2, s_3)$. Variables x_1, x_2 are nonbasic. The solution corresponds to vertex O (the origin). Assuming that x_1 becomes basic, we are increasing the value of x_1, i.e., we are

moving on segment $O - R$. Variables s_1 and s_3 decrease as x_1 increases. The first one to become 0 is s_3. We reach vertex R, where $\underline{x}_B = (x_1, s_1, s_2)$. The next variable to become basic is x_2. We are moving on segment $R - Q$, and obtain the next solution in Q, where $\underline{x}_B = (x_1, x_2, s_2)$. Then s_3 becomes basic and s_2 becomes nonbasic, i.e., we move on segment $Q - P$, reaching the unique optimal vertex P, where $\underline{x}_B = (x_1, x_2, s_3)$,

4. The vertex given by (1) $\cap$ (2) is$P$. That given by (1) $\cap$ (3) is Q. The reduced costs are $\underline{\bar{c}} = \underline{c}^T - \underline{c}_B^T B^{-1} A$. Observe that $\underline{\bar{c}}$ is zero for basic variables, and (possibly!) nonzero for nonbasic ones. Therefore, we only consider $\underline{\bar{c}}_N = \underline{c}_N^T - \underline{c}_B^T B^{-1} N$. In vertex P, we have B and N as in (b)i, so

$$B^{-1}N = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ 2 & 2 \\ 1 & 3 \end{pmatrix}$$

where $\underline{c}'_B = (3, 2, 0)$ and $\underline{c}'_N = (0, 0)$. We obtain $\underline{\bar{c}}_N = (-\frac{7}{4}, -\frac{1}{4})$. Since both values are ≤ 0 and the problem is maximization one, the basic solution corresponding to vertex P is optimal. In vertex Q, we have B, N as in 4.2(b)ii, so

$$B^{-1}N = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 4 \end{pmatrix}$$

where $\underline{c}'_B = (3, 2, 0)$ and $\underline{\bar{c}}_N = (-\frac{5}{3}, \frac{1}{3})$. Therefore, Q is not an optimal solution.

5. The gradient of the objective function is a conic combination of the gradients of the active constraints only in an optimal vertex. It means that any improving direction is infeasible, which implies that the current vertex is optimal. Consider $P = (\frac{1}{2}, 3)$. The gradient of the objective function is $\nabla f = (3, 2)$. The active constraints, in P, are (1) and (2). The gradients are $(2, 1)$ and $(-2, 1)$. We are to check whether the system

$$\lambda_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

admits a solution where $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$. Since it is of full rank, the solution is unique. Since it amounts to $\lambda_1 = \frac{7}{4}, \lambda_2 = \frac{1}{4}$, P satisfies the condition. See Figure 4.

We verify that the condition is not satisfied in the nonoptimal vertices Q, R, O, S.

- Vertex Q. Active constraints (1), (3) with gradients $(2, 1)$, $(1, -1)$. We obtain $\lambda_1 = \frac{5}{3}$, $\lambda_2 = -\frac{1}{3} < 0$. The condition is not satisfied.
- Vertex R. Active constraints (3), $-x_2 \leq 0$ with gradients $(1, -1)$, $(0, -1)$. We obtain $\lambda_1 = 3, \lambda_2 = -5 < 0$. The condition is not satisfied.
- Vertex O. Active constraints $-x_1 \leq 0$, $-x_2 \leq 0$ with gradients $(-1, 0)$, $(0, -1)$. We obtain $\lambda_1 = -3 < 0, \lambda_2 = -2 < 0$. The condition is not satisfied.
- Vertex S. Active constraints $-x_1 \leq 0$, (2) with gradients $(-1, 0)$, $(-2, 1)$. We obtain $\lambda_1 = -7 < 0, \lambda_2 = 2$. The condition is not satisfied.

6. *Geometrical solution, without sensitivity analysis.*

For $b_1 \to \infty$, the optimality of the optimal basis is preserved. Let $S = (0, s)$. If b_1 decreases, while $b_1 > s$, the optimality of the basis is preserved. If $b_1 = s$, we have a

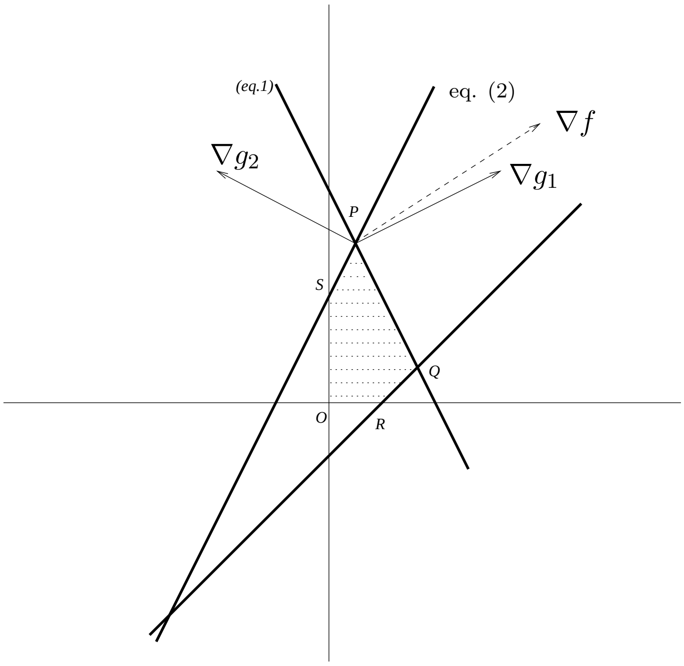

Figure 4: Optimality of vertex P

degenerate solution which can be expressed both as $x_B = (x_1, s_3, x_2)$ or $x_B = (x_1, s_3, s_2)$. For $0 < b_1 < s$, $x_1 = 0$ becomes nonbasic and s_2 becomes basic, since the corresponding constraints (2) is no more active. For $b_1 = 0$, the problem has a single feasible point, $(0, 0)$, and for $b_1 < 0$ the feasible region is empty.

7. S is an optimal solution for any objective function with a gradient which is a conic combination of $(-1, 0)^T$ and $(-2, 1)^T$.
8. The feasible region contains a single point, Q, for b_2 such that Q is the intersection of the three lines corresponding to the three constraints, i.e., for $b_2 = -\frac{8}{3}$. The feasible region is empty for $b_2 < -\frac{8}{3}$.
9. We have multiple optimal solutions if the gradient of the objective function is parallel to that of an inequality defining a facet f of the polyhedron, i.e., if the two gradients are equivalent up to a positive multiplicative factor.

4.3 Simplex method with Bland's rule

The problem is already in standard form. Constraints (5)-(6) equal to the system $A\underline{x} = \underline{b}$, where $\underline{x} = (x_1, x_2, x_3)^T$, $\underline{b} = (1, 5)^T$ and

$$A = \begin{pmatrix} 2 & 0 & 3 \\ 3 & 2 & -1 \end{pmatrix}.$$

Since a feasible basic solution is not evident, we apply the two-phase simplex method.

PHASE I

The goal of the first phase is to find an initial basic feasible solution of the original LP problem, if it exists, or to establish that it is infeasible. This is achieved by solving the following auxiliary LP problem

$$\begin{aligned} \min \quad & v = \sum_i y_i \\ s.t. A\underline{x} + I\underline{y} &= \underline{b} \\ \underline{x} &\geq \underline{0} \\ \underline{y} &\geq \underline{0}, \end{aligned}$$

where an artificial variable y_i is introduced for each equality constraint. If all the components of $\underline{b}$ are nonnegative, the basic solution where y_i, with $1 \leq i \leq m$, are the basic variables and x_j, with $1 \leq j \leq n$, are the nonbasic variables, is an initial basic feasible solution for the auxiliary problem. In order to guarantee that such an initial basic solution is feasible, it suffices for any negative b_i to multiply both sides of the corresponding (i-th) equality constraint by -1.

Since $y_i \geq 0$ for all i with $1 \leq i \leq m$, the objective function value v is greater or equal to 0 and the auxiliary LP problem admits an optimal solution of value $v^* = 0$ if and only if the original LP problem admits a feasible solution. Indeed, to achieve a value of $v = 0$, all variables y_i must be zero, which implies that $A\underline{x} = \underline{b}$ with $\underline{x} \geq \underline{0}$ is feasible. If $v^* > 0$ for the auxiliary LP problem, the original LP problem is infeasible and the two-phase simplex algorithm stops.

For the LP under consideration, the auxiliary LP problem reads $\min\{y_1 + y_2 \mid \bar{A}\bar{x} = b, x \geq 0, y \geq 0\}$. The initial basic feasible solution is $\bar{x}_B = (y_1, y_2)^T$. We express the basic variables w.r.t. the nonbasic ones

$$\begin{aligned} y_1 &= 1 - 2x_1 - 3x_3 \\ y_2 &= 5 - 3x_1 - 2x_2 + x_3 \end{aligned}$$

so that the objective function becomes $v = y_1 + y_2 = 6 - 5x_1 - 2x_2 - 2x_3$. The initial tableau is

		x_1	x_2	x_3	y_1	y_2
	-6	-5	-2	-2	0	0
y_1	1	2	0	3	1	0
y_2	5	3	2	-1	0	1

The reduced costs of the nonbasic variables $\bar{x}_N = (x_1, x_2, x_3)^T$ are all negative. According to Bland's rule, we pick as nonbasic variable to enter the basis the variable with smallest index, x_1. There are two limitations to the growth of x_1, i.e., given by $y_1 = 1 - 2x_1$ and $y_2 = 5 - 3x_1$. Therefore, the tightest upper limit is $\min\{\frac{1}{2}, \frac{5}{3}\} = \frac{1}{2}$, given by the nonnegativity of the basic variable y_1, which leaves the basis. Pivoting is performed on coefficient 2 in tableau position (1,1)

1. divide row 1 by 2
2. add 5 times row 1 to row 0
3. subtract 3 times row 1 from row 2

We obtain the tableau

		x_1	x_2	x_3	y_1	y_2
	$-\frac{7}{2}$	0	-2	$\frac{11}{2}$	$\frac{5}{2}$	0
x_1	$\frac{1}{2}$	1	0	$\frac{3}{2}$	$\frac{1}{2}$	0
y_2	$\frac{7}{2}$	0	2	$-\frac{11}{2}$	$-\frac{3}{2}$	1

which shows the current basic solution $\underline{x}_B = (x_1, y_2)^\intercal = (1/2, 7/2)^\intercal$ of value 7/2 (obviously $\underline{x}_N = 0$). Since the only negative reduced cost is given by x_2, x_2 enters the basis. The only limit to its growth is given by $y_2 = \frac{7}{2} - 2x_2$, therefore $x_2 \leq \frac{7}{4}$ and y_2 leaves the basis. Pivoting is performed on coefficient 2 in position (2,2) of the previous tableau

1. add row 2 to row 0
2. divide row 2 by 2

We obtain

		x_1	x_2	x_3	y_1	y_2
	0	0	0	0	1	1
x_1	$\frac{1}{2}$	1	0	$\frac{3}{2}$	$\frac{1}{2}$	0
x_2	$\frac{7}{4}$	0	1	$-\frac{11}{4}$	$-\frac{3}{4}$	$\frac{1}{2}$

Since the reduced costs for the nonbasic variables are all nonnegative, Phase I stops, proving that the feasible region of the original problem is nonempty, and yielding an initial basic feasible solution with the basic variables $\underline{x}_B = (x_1, x_2)^T$.

PHASE II

The objective function of the original LP is $z = x_1 - 2x_2$, the initial basic variables are $\underline{x}_B = (x_1, x_2)^T$ and the only nonbasic variable is x_3. We express the basic variables x_1, x_2 w.r.t. the nonbasic variable x_3

$$x_1 = \frac{1}{2} - \frac{3}{2}x_3$$
$$x_2 = \frac{7}{4} + \frac{11}{4}x_3$$

The objective function becomes $z = -3 - 7x_3$. Removing from the tableau the columns of the auxiliary variables, we obtain

		x_1	x_2	x_3
	3	0	0	-7
x_1	$\frac{1}{2}$	1	0	$\frac{3}{2}$
x_2	$\frac{7}{4}$	0	1	$-\frac{11}{4}$

The variable x_3, having a negative reduced cost of -7, enters the basis. Since the only limit to its growth is given by $x_1 = \frac{1}{2} - \frac{3}{2}x_3$, x_1 leaves the basis. Therefore, we perform a pivoting operation on coefficient $\frac{3}{2}$ in position (3,1)

1. divide row 1 by $\frac{3}{2}$;

2. add 7 times row 1 to row 0
3. add $-\frac{11}{4}$ times row 1 to row 3

We obtain

		x_1	x_2	x_3
	$\frac{16}{3}$	$\frac{14}{3}$	0	0
x_3	$\frac{1}{3}$	$\frac{2}{3}$	0	1
x_2	$\frac{8}{3}$	$\frac{11}{6}$	1	0

Since the reduced costs are all nonnegative, the algorithm terminates and it yields the optimal solution $(0, \frac{8}{3}, \frac{1}{3})^T$ with an objective function value $z^* = -\frac{16}{3}$.