4.1 Graphical solution of a linear program and standard form

Consider the problem

$$\begin{aligned} \min \quad & \underline{c}^T \underline{x} \\ & A\underline{x} \geq \underline{b} \\ & \underline{x} \geq \underline{0} \end{aligned}$$

where

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \underline{c} = \begin{pmatrix} 16 \\ 25 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 4 \\ 5 \\ 9 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 7 \\ 1 & 5 \\ 2 & 3 \end{pmatrix}$$

1. Solve the problem graphically and determine the basic and nonbasic variables of the optimal solution.
2. Put the problem in standard form w.r.t. the optimal basis (identify B, N, and the corresponding partition of the cost vector).

4.2 Geometry of linear programming

Consider the linear program

$$\begin{aligned} \max \quad z = \ & 3x_1 + 2x_2 \\ & 2x_1 + x_2 \leq 4 && (1) \\ & -2x_1 + x_2 \leq 2 && (2) \\ & x_1 - x_2 \leq 1 && (3) \\ & x_1, x_2 \geq 0 \end{aligned}$$

1. Solve it graphically and indicate the optimal solution and the corresponding objective function value.
2. Determine the basic feasible solutions (specifying the basic and nonbasic variables) corresponding to all vertices of the feasible region (polyhedron).
3. Indicate the sequence of basic feasible solutions that are visited by the simplex algorithm (let x_1 be the first variable to enter the basis) when starting from the initial basic feasible solution in which all slack variables are basic variables.
4. Determine the reduced costs for the basic feasible solutions associated to the vertices ((eq. 1) $\cap$ (eq. 2)) and ((eq. 1) $\cap$ (eq. 3)), where (eq. i) is obtained from (i), by substituting $\leq$ with $=$.
5. Show, geometrically, that the gradient of the objective function is a conic combination (i.e. a linear combination with nonnegative coefficients) of the gradients of the constraints which are active at an optimal vertex. Indicate the value taken by the objective function in that vertex. *Note*: All the constrains must be in $\leq$ form, since the problem is a maximization one (e.g., $x_1 \geq 0$ must be rewritten as $-x_1 \leq 0$).

6. Determine the range of values for the right hand side b_1 of constraint (1) for which the optimality of the basis solution is preserved.

7. Indicate for which values of the objective function coefficients $((x_1 = 0) \cap$ (eq. 2)) is an optimal vertex.

8. Determine the range of values for the right hand side b_2 of constraint (2) for which the feasible region is (a) empty, (b) contains a single point.

9. Indicate the range of values for c_1 for which there are multiple optimal solutions.

4.3 Simplex algorithm with Bland's rule

Solve the following linear program

$$\min \quad z = x_1 - 2x_2 \tag{4}$$
$$2x_1 + 3x_3 = 1 \tag{5}$$
$$3x_1 + 2x_2 - x_3 = 5 \tag{6}$$
$$x_1, x_2, x_3 \geq 0 \tag{7}$$

by applying the two-phase Simplex algorithm with Bland's rule.