Exercise 2.8

Application of Dynamic Programming

A company buys a new machine for 12 K€.

Annual maintenance costs for the next five years:

Age (years)	Maintenance (K€)	Selling price (K€)
0	2	-
1	4	7
2	5	6
3	9	2
4	12	1

To avoid high maintenance costs of an older machine, the machine can be replaced at the beginning of the 2^{nd}, 3^{rd}, 4^{th} or 5^{th} year with a new machine.

To simplify the model we assume that the cost of a new machine is constant (12 K€).

Objective: minimize the *net total cost* (price + maintenance – selling price) over the 5 year horizon.

Age	Main.	Sell.
0	2	-
1	4	7
2	5	6
3	9	2
4	12	1

If we buy a machine at beginning of 1st year and sell it at the beginning of 2nd year, the net cost is: 12 + 2 - 7 = 7

If we buy it at the beginning of 1st year and sell it at the beginning of 3rd year, the net cost is:12 + 2 + 4 - 6 = 12

etc...

Show how the problem of determining an optimal maintenance-replacement plan of minimum total cost (for the next 5 years) can be solved via Dynamic Programming.

Hint: Reduce it to a minimum cost path problem in an appropriate acyclic graph.

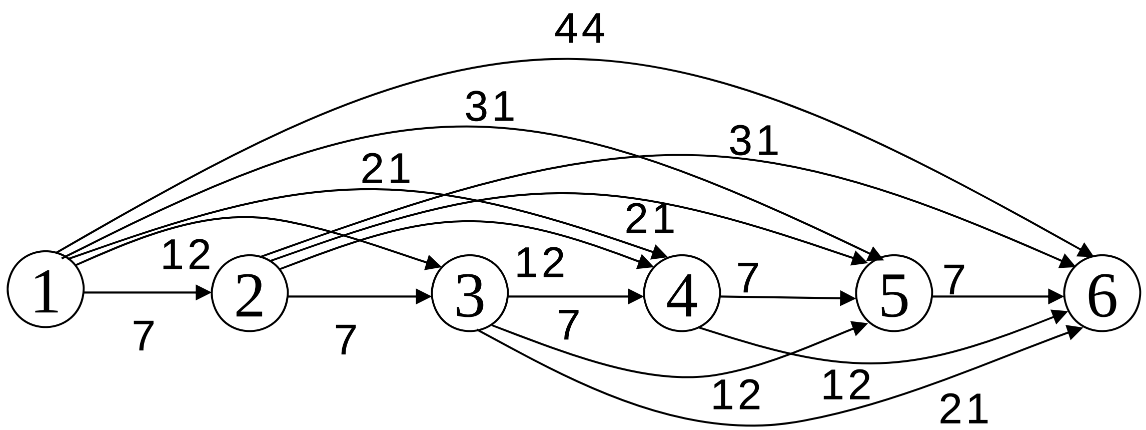
Find all the optimal maintenance-replacement plans.

Model:

Associate a node to the beginning of each year (node *i* at the beginning of the *i*-th year).

An arc (*i*, *j*) represents the choice of buying a machine at the beginning of the *i*-th year and selling it at the beginning of the *j*-th year.

age	main	rec.
0	2	-
1	4	7
2	5	6
3	9	2
4	12	1

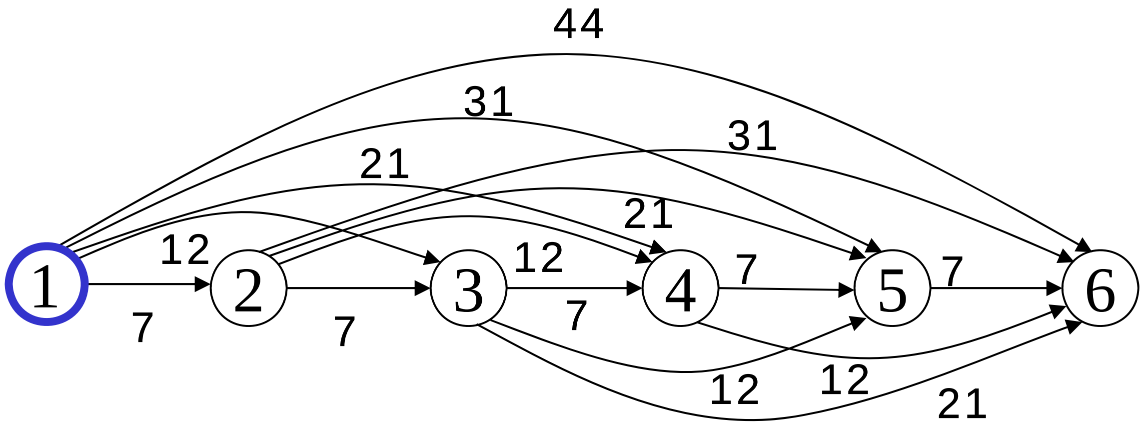


c_{ij} = cost to buy a machine at the beginning of the *i*-th year
+ maintenance costs during the years *i*, *i*+1,…, *j*-1
- selling price of the old machine at the beginning of *j*-th year

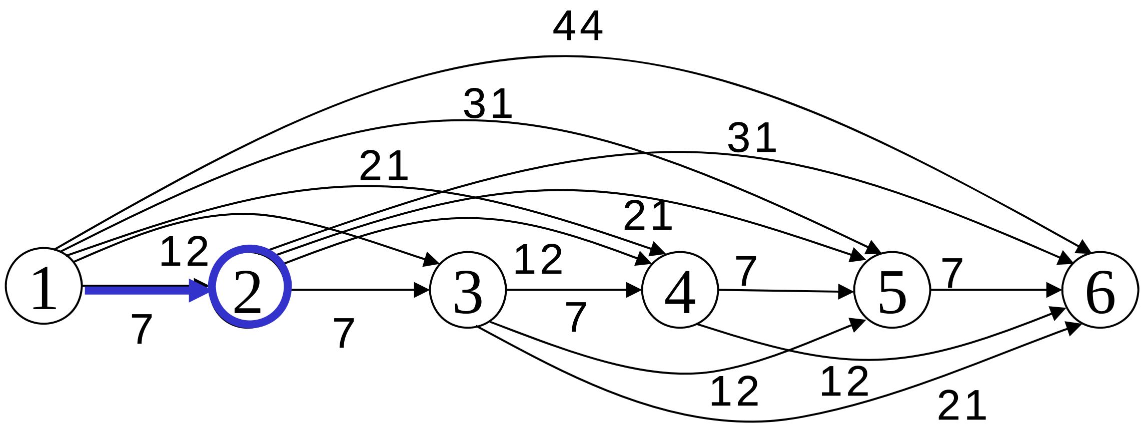
Any path from node 1 to node 6 corresponds to a maintenance and replacement plan for the machine and its cost amounts to the net total cost of this plan during the 5 years.

Since we wish to minimize the total net cost, we have to determine a minimum cot path from nodes 1 to 6.

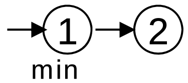
Since the graph *G* is acyclic we can apply Dynamic Programming (the nodes are already topologically ordered).



$L(1) = 0$ [-]



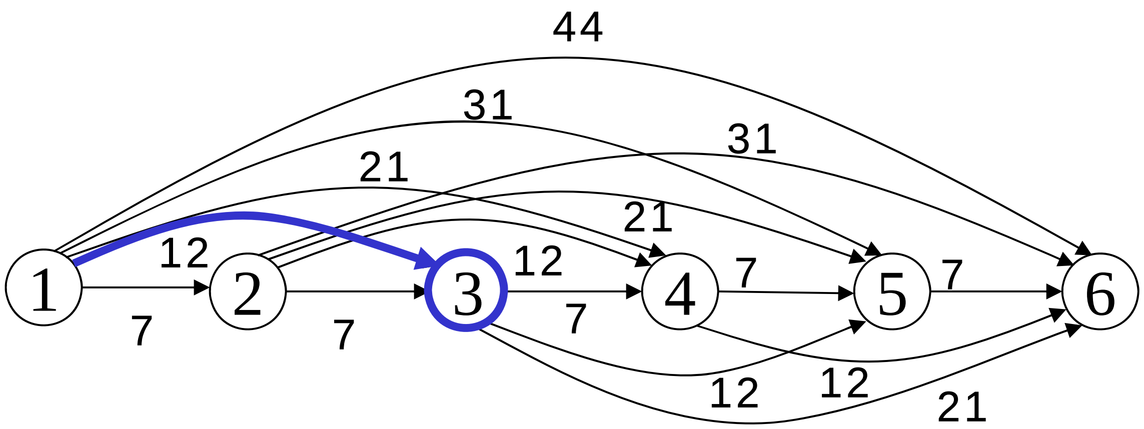
Entering arcs in node ②:



$L(1) + c_{12} =$

$0 + 7 = 7$

$L(2) = \min \{ 7 \} = 7$ [1]

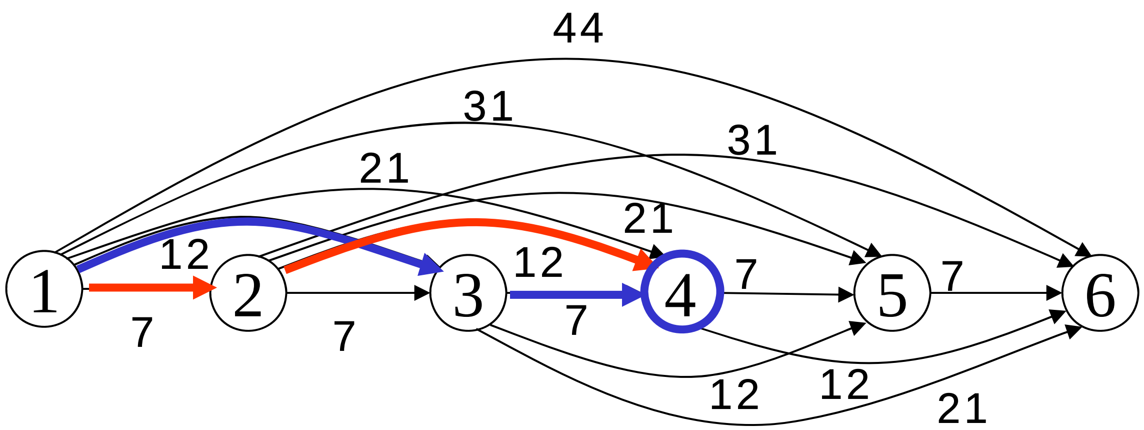


Entering arcs in node ③:

→①→③ (min) L(1) + 12 = 0 + 12 = 12 ←

→②→③ (min) L(2) + 7 = 7 + 7 = 14

L(3) = min { 12, 14 } = 12 [1]



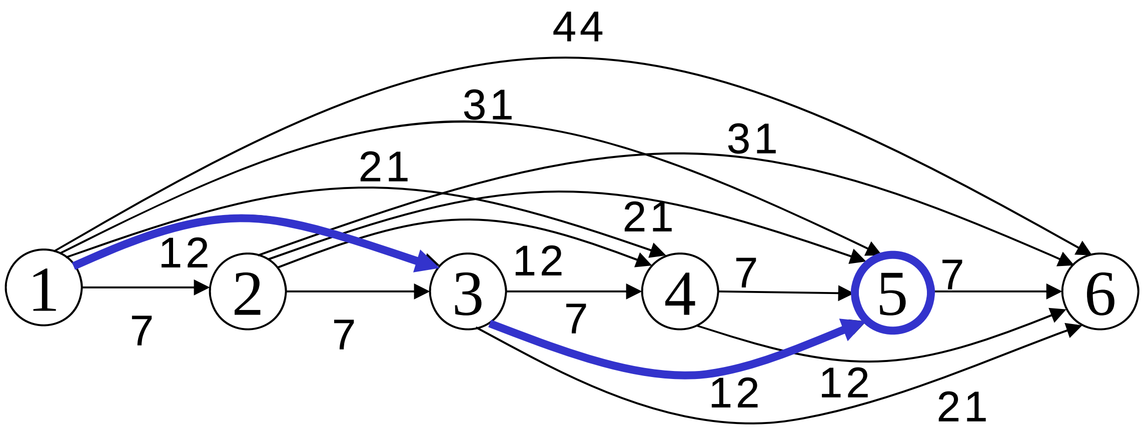
Entering arcs in node ④:

→①→④ (min) L(1) + 21 = 0 + 21 = 21

→②→④ (min) L(2) + 12 = 7 + 12 = 19 ←

→③→④ (min) L(3) + 7 = 12 + 7 = 19 ←

L(4) = min { 21, 19, 19 } = 19 [3]



Entering arcs in node 5 :

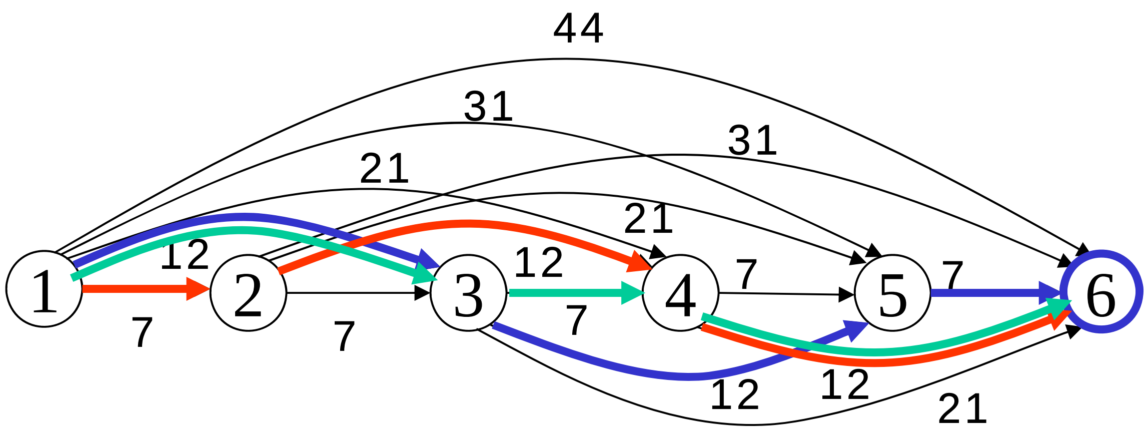
→(1)→(5) min — L(1) + 31 = 0 + 31 = 31

→(2)→(5) min — L(2) + 21 = 7 + 21 = 28

→(3)→(5) min — L(3) + 12 = 12 + 12 = 24 ←

→(4)→(5) min — L(4) + 7 = 19 + 7 = 26

L(5) = min { 31, 28, 24, 26 } = 24 [3]



Entering arcs in node 6 :

→(1)→(6) min — 0 + 44 = 44

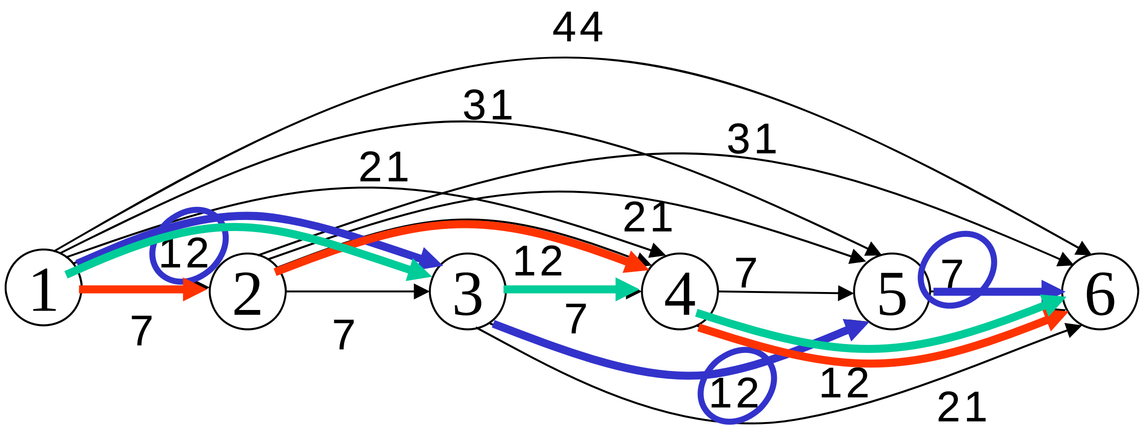
→(2)→(6) min — 7 + 31 = 38

→(3)→(6) min — 12 + 21 = 33

→(4)→(6) min — 19 + 12 = 31 ←

→(5)→(6) min — 24 + 7 = 31 ←

L(6) = min { 44, 38, 33, 31, 31 } = 31 [5]



minimum cost = <u>31</u>

Cost of a shortest path from (1) a (6) = minimum <u>net cost</u> of the corresponding <u>maintenance and replacement plan</u> during the 5 years.