1.8 Modem production

A company A, producing a single type of modem, is planning the production for the next 4 months. The production capacity is of 2500 units per month, at a cost of 7 Euro per unit. A second company B can be payed to produce extra units, at a cost of 9 Euro per unit, for a maximum of 700 units per month.

Inventory can be used, at a cost of 1 Euro per month per unit. 70 units are stored at the beginning of the planning period, and 300 units must be stored at the end of the period.

For technical reasons, the units must be produced in lots of, at least, 80 units. Sales forecasts for the next 4 months indicate the following demand

Month	1	2	3	4
Demand	2600	3000	2350	2700

Give an Integer Linear Programming (ILP) formulation for the problem of determining a production plan of minimum total cost.

Suppose that fixed charges f_A and f_B must be payed each month that production takes place in factory A and, respectively, B. How can we extend the ILP formulation to account for fixed charges?

1.9 Hospital shifts

We have to plan the shifts for a hospital department, so that the total number of nurses is minimized. Each nurse works for 5 consecutive days, and stays home for the next two days. The estimated demand, in term of number of nurses, is

Day	Mo	Tu	We	Th	Fr	Sa	Su
Demand	11	9	7	12	13	8	5

Give an integer linear programming formulation for the problem.

1.10 Cash flow

A company expects the following cash flow (in kEuro) for the next 8 months

Month	inflow	outflow
May	100	500
June	200	500
July	200	600
...	...	...
Dec	900	100

At the beginning of the planning period (May), we have 100K Euro of cash. Money can be loaned from banks, with two options: 1) loan for 8 months, with 9% interest overall, or 2) loan for 1 month, with 1.5% interest per month. For instance, with option 1, if 100 Euro are loaned at the beginning of May, 109 Euro must be given back at the end of December; with option 2, if 100 Euro are loaned at the beginning of month i, 101.5 Euro must be given back at the begining of month $(i + 1)$.

No debt must be estabilished after the end of the planning period. A cash of 50k Euro is required at the end of the period.

Give a linear programming formulation for the problem of minimizing the loaning cost.

How should we modify the formulation if only one of the two loaning options can be chosen?

SOLUTION

1.8 Modem production

Sets

- J: production months

Parameters

- d_j: demand for month j
- c_A: unit production cost for company A
- c_B: unit production cost for company B
- m: unit inventory cost per month
- l: minimum lot size
- u_A: maximum number of units in production per month for company A
- u_B: maximum number of units in production per month for company B

Variables

- x_{Aj}: units of product produced in month $j \in J$ by company A
- x_{Bj}: units of product produced in month $j \in J$ by company B
- y_{Aj}: 1 if company A produces in month $j \in J$, 0 otherwise
- y_{Bj}: 1 if company B produces in month $j \in J$, 0 otherwise
- z_j: units in inventory at the end of month $j \in J$

Model

$$
\begin{array}{lllr}
\min & c_A \sum_{j=1}^{4} x_{Aj} + c_B \sum_{j=1}^{4} x_{Bj} + m \sum_{j=0}^{4} z_j & & \\
s.t. & x_{Aj} + x_{Bj} + z_{j-1} \geq d_j & j \in J & \text{(demand)} \\
& z_j = z_{j-1} + x_{Aj} + x_{Bj} - d_j & j \in J & \text{(inventory balance)} \\
& z_0 = 70 & & \\
& z_4 = 300 & & \\
& l y_{Aj} \leq x_{Aj} \leq u_A y_{Aj} & & \text{(min lot size)} \\
& l y_{Bj} \leq x_{Bj} \leq u_B y_{Bj} & & \text{(min lot size)} \\
& x_{Aj}, x_{Bj}, z_j \in \mathbb{N} & j \in J & \\
& y_j \in \{0,1\} & j \in J &
\end{array}
$$

Observe that constraints *(demand)* are redundant, since they are implied by the *(inventory balance)* constaints and the nonnegativity of z_j.

For the variant with fixed costs, we introduce into the objective function the additional term

$$f_A \sum_{j=1}^{4} y_{Aj} + f_B \sum_{j=1}^{4} y_{Bj}.$$

1.9 Hospital shifts

Let the variables x_0, x_1, x_2, x_3, x_4, x_5, and x_6 denote the number of nurses whose shift begins, respectively, on Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, and Sunday. We have the following integer linear programming model

$$\begin{array}{llllllllll}
\min & (x_0 & +\, x_1 & +\, x_2 & +\, x_3 & +\, x_4 & +\, x_5 & +\, x_6) & & \\
& & & & & & & & & \\
& x_0 & & & +\, x_3 & +\, x_4 & +\, x_5 & +\, x_6 & \geq 11 & \text{(Mo)} \\
& x_0 & +\, x_1 & & & +\, x_4 & +\, x_5 & +\, x_6 & \geq 9 & \text{(Tu)} \\
& x_0 & +\, x_1 & +\, x_2 & & & x_5 & +\, x_6 & \geq 7 & \text{(We)} \\
& x_0 & +\, x_1 & +\, x_2 & +\, x_3 & & & +\, x_6 & \geq 12 & \text{(Th)} \\
& x_0 & +\, x_1 & +\, x_2 & +\, x_3 & +\, x_4 & & & \geq 13 & \text{(Fr)} \\
& & x_1 & +\, x_2 & +\, x_3 & +\, x_4 & +\, x_5 & & \geq 8 & \text{(Sa)} \\
& & & x_2 & +\, x_3 & +\, x_4 & +\, x_5 & +\, x_6 & \geq 5 & \text{(Su)}
\end{array}$$

$$x_0, x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbb{Z}^+$$

Optional: We generalize the model as follows

Sets

- J: days

Parameters

- d: days for which a nurse works (5)
- $w = |J|$: days in a period (7)

Variables

- x_j: number of nurses starting their "5 day week" on day $j \in J$

Model

$$
\begin{aligned}
&\min \sum_{j=0}^{w-1} x_j \\
&s.t. \sum_{j=0}^{d-1} x_{mod(w+(i-j),w)} \geq d_i && i \in J \\
&x_j \in \{0,1\} && j \in J
\end{aligned}
$$

To assess the correctness, try to write the constraints for, e.g., $i = 2$ (We). We get

j	$x_{mod(7+(2-j),7)}$
0	mod(7+2,7) = 2
1	mod(7+1,7) = 1
2	mod(7+0,7) = 0
3	mod(7-1,7) = 6
4	mod(7-2,7) = 5

1.10 Cash flow

Sets

- $I = \{1, \ldots, 8\}$: months

Parameters

- e_i: inflow at month $i \in I$
- u_i: outflow at month $i \in I$

Variables

- x: amount of money loaned at month 1 with option 1
- y_i: amount of money loaned at month $i \in I \setminus \{8\}$ with option 2
- z_i: cash at month $i \in I$

We cannot establish debts that are to be estinguished after month 8. Therefore, we define variable x only for month 1, and variables y_i only for months $\{1, \ldots, 7\}$.

Model

$$
\begin{aligned}
\min \quad & 0.09x + \sum_{i=1}^{7} 0.015 y_i \\
\text{s.t.} \quad & \\
& z_i = z_{i-1} + y - (1 + 0.015) y_{i-1} + e_i - u_i \quad i \in \{2, \dots, 6\} \\
& z_1 = z_0 + y_1 + x + e_1 - u_1 \\
& z_8 = z_7 - (1 + 0.015) y_7 - x - (1 + 0.09) x + e_8 - u_8 \\
& z_0 = 80 \\
& z_8 \geq 50 \\
& x, y_i, z_i \geq 0 \quad i \in I
\end{aligned}
$$

For the variant, we introduce the variable k and the constraints

$$
\begin{aligned}
y_i &\leq Mk, i \in \{1, \dots, 7\} \\
x &\leq M(1 - k)
\end{aligned}
$$

where M is a large enough value, larger than the largest value that x, y_i can take.