1.5 Computer rental planning

A company must rent computers in order to face the requirements during the next four months:

Month	January	February	March	April
Requirements	9	5	7	9

The rental cost depends on the length of the rental as follows:

Length	1 month	2 months	3 months
Cost (Euro)	200	350	450

Give an integer linear programming formulation for the problem of finding a rental plan of minimum total cost.

1.6 Workforce planning

A ICT services company has estimated the following demand for maintenance and consultancy for the next 5 months:

Month	1	2	3	4	5
Demand (hours)	6000	7000	8000	9500	11000

At the beginning of the planning period, 50 technicians are available. Each technician works at most 160 hours per month. To satisfy the demand, new technicians must be hired and trained. During the training period, which lasts one month, an expert technician must coach the newly hired one for 50 hours.

Expert technicians are payed 2000 Euro per month, while the newly hired ones are payed 1000 Euro, during the training month. We suppose that, at the end of each month, 5% of the expert technicians leave the company to join competitors.

Give a mathematical programming formulation for the problem of minimizing the total costs.

1.7 Facility location and transportation

A distribution company has to supply a single type of good to n clients. Let d_j denote the demand of client j, where $j \in J = \{1, \dots n\}$. The goods must be stocked in warehouses before being delivered to the clients. The warehouses can be located in m candidate sites. Let f_i denote the cost for opening a warehouse at the candidate site $i \in I$, with $i \in I = \{1, \dots, m\}$, and b_i denote its maximum capacity. Let c_{ij} be the unit transportation cost from warehouse i to client j, and q_{ij} be the maximum quantity to be transported from warehouse i to client j.

Give an integer linear programming formulation to decide where to locate the warehouses and how to satisfy the client demands so as to minimize the total costs, i.e., the total opening costs plus the total transportation costs.

SOLUTION

1.5 Computer rental planning

Variables

- g_1, g_2, g_3: number of computers rented in January for 1, 2, and 3 months
- f_1, f_2, f_3: number of computers rented in February for 1, 2, and 3 months
- m_1, m_2: number of computers rented in March for 1 and 2 months
- a_1: number of computers rented in April for 1 month

Model

$$
\begin{array}{lll}
\min & 200(g_1 + f_1 + m_1 + a_1) & \\
 & + 350(g_2 + f_2 + m_2) + 450(g_3 + f_3) & \text{(cost)} \\
\text{s.t.} & & \\
 & g_1 + g_2 + g_3 \geq 9 & \text{(January)} \\
 & f_1 + f_2 + f_3 + g_2 + g_3 \geq 5 & \text{(February)} \\
 & m_1 + m_2 + f_2 + f_3 + g_3 \geq 7 & \text{(March)} \\
 & a_1 + m_2 + f_3 \geq 9 & \text{(April)} \\
 & g_1, g_2, g_3, f_1, f_2, f_3, m_1, m_2, a_1 \in \mathbb{Z}^+ & \text{(integer nonnegative variables)}
\end{array}
$$

We can generalize by introducing the following indices

Sets

- $I = \{1, \ldots, m\}$: months
- $J = \{1, \ldots, l\}$: rental lengths (in months)

Parameters

- d_i: demand for month i, $i \in I$
- c_j: cost for rental of length j, $j \in J$

Variables

- x_{ij}: number of computers rented in month i for j months, $i \in I, j \in J$

Model

$$\begin{array}{lll}
\min & \displaystyle\sum_{i\in I, j\in J} c_j x_{ij} & \text{(cost)} \\
\text{s.t.} & & \\
& \displaystyle\sum_{k=\max\{1,i-l+1\}}^{i} \sum_{j=i-k+1}^{l} x_{kj} \geq d_i \quad\quad i \in I & \text{(demand)} \\
& x_{ij} \in \mathbb{Z}^+ \quad\quad i \in I, j \in J & \text{(nonnegative integer variables)}
\end{array}$$

Constraint *(demand)* works as follows. The initial value for k is such that, if we can rent for, e.g., at most $l = 3$ months, when, e.g., $i = 7$, k will start from $7 - 3 + 1 = 5$. The min operator grants $k \geq 1$ (consider, for instance, $i = 1, l = 3$, where we would have $i-l+1 = 1-3+1 = -1$). With the second summation we take into account the fact that if, e.g., $i = 3$, $k = 2$, and $l = 3$, then j must be at least 2, otherwise, with $j = 1$, we would consider the variable x_{21} in the summation, which is wrong. Indeed, it amounts to consider, in month 3, machines which have been rented for one month in month 2, which are no more available in month 3.

1.6 Workforce planning

Sets

- $I = \{1, \ldots, 5\}$: months

Parameters

- d_i: demand for month i, $i \in I$

Variables

- x_i: number of expert technicians available in month i, $i \in I$
- y_i: number of technicians in training available in month i, $i \in I$

Model

$$\begin{array}{lll}
\min & 2000\,(x_1 + x_2 + x_3 + x_4 + x_5) + 1000\,(y_1 + y_2 + y_3 + y_4 + y_5) & \\
\text{s.t.} & & \\
& 160x_i - 50y_i \geq d_i & i \in I \quad \text{(demand)} \\
& \lfloor 0.95x_i \rfloor + y_i = x_{i+1} & i \in I \quad \text{(number)} \\
& x_1 = 50 & \\
& x_i, y_i \in \mathbb{Z}^+ \quad i \in I \quad \text{(nonnegative integer variables)} &
\end{array}$$

Obviously, Constraint (number) is nonlinear. By dropping the $\lfloor \cdot \rfloor$ operator, we obtain the constraint

$$0.95x_i + y_i = x_{i+1} \quad i \in I$$

which is not correct since, due to the integrality requirements on the variables, it forces x_i to be a multiple of 100 (such that $0.95x_i$ will always be an integer value).

Constraint (number) can be formulated in a linear way by introducing for each $i \in I$ an auxiliary integer variable z_i that represents $\lfloor 0.95x_i \rfloor$, and by adding the auxiliary constraints:

$$\begin{array}{llr} z_i + y_i = x_{i+1} & i \in I & \text{(number2)} \\ 100z_i \leq 95x_i \leq 100z_i + 100 & i \in I & \text{(floor)} \\ z_i \in \mathbb{Z}^+ \quad i \in I. \qquad \text{(nonnegative integer variables)} & & \end{array}$$

Then we clearly have $z_i = \lfloor 0.95x_i \rfloor$.

1.7 Facility location and transportation

The problem is an important extension of the *transportation problem*, where we also have to decide the location of the warehouses from which the goods are shipped.

Variables

- x_{ij}: amount of good to be transported from warehouse $i \in I$ to client $j \in J$
- y_i: 1 if warehouse $i \in I$ is active, and 0 otherwise

Model

$$\begin{array}{llr} \min \displaystyle\sum_{i \in I, j \in J} c_{ij}x_{ij} + \sum_{i \in I} f_i y_i & & \text{(total cost)} \\ \text{s.t.} \displaystyle\sum_{j \in J} x_{ij} \leq b_i y_i & i \in I & \text{(availability)} \\ \displaystyle\sum_{i \in I} x_{ij} \geq d_j & j \in J & \text{(demand)} \\ 0 \leq x_{ij} \leq q_{ij} & i \in I, j \in J & \text{(bounds)} \\ y_i \in \{0, 1\} & i \in I & \text{(binary vars)} \end{array}$$

Since $x_{ij} \leq q_{ij}$, for all $i \in I$ and $j \in J$, we can substitute

$$x_{ij} \leq q_{ij} \qquad i \in I, j \in J \quad \text{(bound)}$$

with

$$x_{ij} \leq q_{ij} y_i \qquad i \in I, j \in J \quad \text{(quantity 2)}$$

and we obtain the alternative formulation:

$$
\begin{array}{llr}
\min \sum_{i\in I, j\in J} c_{ij}x_{ij} + \sum_{i\in I} f_i y_i & & \text{(total cost)}\\
\text{s.t.} \sum_{j\in J} x_{ij} \leq b_i & i \in I & \text{(availability 2)}\\
\sum_{i\in I} x_{ij} \geq d_j & j \in J & \text{(demand)}\\
0 \leq x_{ij} \leq q_{ij} y_i & i \in I, j \in J & \text{(bounds 2)}\\
y_i \in \{0,1\} & i \in I & \text{(binary vars)}
\end{array}
$$

Note that, by summing up all the (quantity 2) constraints for $j \in J$ with a unit multiplier, we obtain

$$\sum_{j\in J} x_{ij} \leq (\sum_{j\in J} q_{ij}) y_i$$

which is tighter than the (availability) constraints, whenever $\sum_{j\in J} q_{ij} < b_i$.

If we take the (availability) constraints for some $i \in I$, fix $j \in J$, and sum them up with $-x_{ij'} \leq 0$ for all $j' \in J \setminus \{j\}$, we obtain

$$x_{ij} \leq b_i y_i$$

which is weaker than the (quantity) constraints whenever $b_i > q_{ij}$.