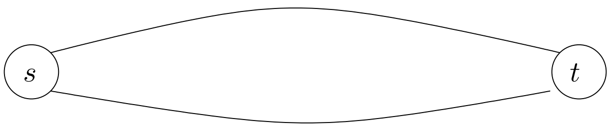
1.3 Packet routing

n packets of data must be routed from node s to node t, along one of two available links, with capacity (bandwidth) $k_1 = 1$ Mbps and $k_2 = 2$ Mbps.



The cost per unit of capacity of link 2 is 30% larger than that of link 1. The following table indicates the quantity of capacity consumed by each packet i, $i \in \{1 \dots n\}$, and the cost to route it on link 1.

Packed	Consumed capacity	Cost on link 1
1	0.3	200
2	0.2	200
3	0.4	250
4	0.1	150
5	0.2	200
6	0.2	200
7	0.5	700
8	0.1	150
9	0.1	150
10	0.6	900

Give an integer linear programming formulation for the problem of minimizing the total cost of routing all the packets. Give also an integer linear programing formulation for the more general case where m links are available.

1.4 Multi-period production planning (uncapacitated lot sizing)

A company A, which produces one type of high-precision measuring instrument, has to plan the production for the next 3 months. Each month, A can produce at most 110 units, at a unit cost of 300 Euro. Moreover, each month, up to 60 additional units produced by another company B can be bought at a unit cost of 330 Euro. Unsold units can be stored. The inventory cost is of 10 Euro per unit of product, per month. Sales forecasts indicate a demand of 100, 130, and 150 units of product for the next 3 months.

1. Give a linear programming formulation for the problem of determinig a production plan (direct or indirect) which minimizes the total costs, while satisfying the monthly demands.
2. Give a mixed integer linear programming formulation for the variant of the problem where production lots have a minimum size. In particular, if any strictly positive quantity is produced in a given month, this quantity cannot be smaller than 15 units.

Solution

1.3 **Packet routing**.

The integer linear programming formulation for the 2-link case is as follows

Sets

- $I = \{1 \dots n\}$: set of packets

Parameters

- a_i: capacity consumed by packet i, for $i \in I$
- c_{i1}, c_{i2}: routing cost for packet i on link 1 and 2, for $i \in I$
- k_1, k_2: capacity for link 1 and 2

Variables

- x_i: 1 if packet i is routed on link 1, 0 if routed on link 2, for $i \in I$

Model

$$\begin{array}{llll} \min & \sum_{i \in I} c_{i1} x_i + c_{i2}(1 - x_i) & & \text{(cost)} \\ \text{s.t.} & & & \\ & \sum_{i \in I} a_i x_i \leq k_1 & & \text{(capacity 1)} \\ & \sum_{i \in I} a_i (1 - x_i) \leq k_2 & & \text{(capacity 2)} \\ & x_i \in \{0, 1\} & i \in I & \text{(binary variables)} \end{array}$$

The m-link formulation requires a new set of binary variables, one for each packet and link. The packet-to-link assignment is also to be explicitly introduced.

Sets

- $I = \{1 \dots n\}$: set of packets
- $J = \{1 \dots m\}$: set of links

Parameters

- a_i: capacity consumed by packet i, for $i \in I$
- c_{ij}: routing cost for packet i on link j, for $i \in I, j \in J$
- k_j: capacity of link j, for $j \in J$

Variables

- x_{ij}: 1 if packet i is routed on link j, 0 otherwise, for $i \in I, j \in J$

Model

$$
\begin{array}{llll}
\min & \sum_{i \in I, j \in J} c_{ij} x_{ij} & & \text{(cost)} \\
\text{s.t.} & & & \\
& \sum_{j \in J} x_{ij} = 1 & i \in I & \text{(assignment)} \\
& \sum_{i \in I} a_i x_{ij} \leq k_j & j \in J & \text{(capacity)} \\
& x_{ij} \in \{0, 1\} & i \in I, j \in J & \text{(binary variables)}
\end{array}
$$

1.4 Multi-period production planning (uncapacitated lot sizing)

Sets

- $T = \{1 \ldots 3\}$: set of months

Parameters

- b: production capacity of A
- b': production capacity of B
- c: unit production cost for A
- c': unit production cost for B
- m: inventory cost per unit and month
- d_t: sales forecasts for month t, for $t \in T$

Variables

- x_t: units produced by A in month t, for $t \in T$
- x'_t: units bought from B in month t, for $t \in T$
- z_t: units in inventory at the end of month t, for $t \in T \cup \{0\}$

Model

$$
\begin{array}{llll}
\min & \sum_{t \in T} (c x_t + c' x'_t + m z_t) & & \text{(cost)} \\
\text{s.t.} & & & \\
& x_t \leq b & t \in T & \text{(capacity of A)} \\
& x'_t \leq b' & t \in T & \text{(capacity of B)} \\
& z_{t-1} + x_t + x'_t \geq d_t & t \in T & \text{(demand)} \\
& z_{t-1} + x_t + x'_t - d_t = z_t & t \in T & \text{(inventory balance)} \\
& z_0 = 0 & & \text{(starting condition)} \\
& x_t, x'_t, z_t \geq 0 & t \in T & \text{(nonnegative variables)}
\end{array}
$$

Observe that the (*demand*) constraint is redundant, as it is implied by $z_{t-1}+x_t+x'_t-d_t = z_t \geq 0$.

To take into account the minimum lot size, we add the binary variables

- y_t: 1 if production is active at month t, 0 otherwise, for $t \in T$

and the constraints

$$[rlll] \quad x_t \geq l y_t \qquad t \in T \qquad \text{(minimum lot size)}$$
$$x_t \leq M y_t \qquad t \in T \qquad \text{(activation)},$$

where $l = 15$ is the minimum lot size, and M is a large enough value, such that constraint $x_t \leq My_t$ is redundant when $y_t = 1$. For instance, we can choose $M = 110$, i.e., equal to the monthly productive capacity. Such constraints are usually called *big-M* constraints.