1.1 Portfolio optimization

A bank has a capital of C billions of Euro and two available stocks:

1. with an annual revenue of 15% and a risk factor of $\frac{1}{3}$,
2. with an annual revenue of 25% and risk factor of 1.

The risk factor represents the maximum fraction of the stock value that can be lost. A risk factor of 0.25 implies that, if stocks are bought for 100 Euro up to 25 Euro can be lost. It is required that at least half of C is risk-free. The amount of money used to buy stocks of (2) must not be larger than two times that used to buy stocks of (1). At least $\frac{1}{6}$ of C must be invested into (1).

Give a Linear Programming formulation for the problem of determining an optimal portfolio for which the profit is maximized. Solve the problem graphically.

1.2 Gasoline mixture

A refinery produces two types of gasoline, mixing three basic oils according to the following gasoline mixture rules:

	Oil 1	Oil 2	Oil 3	Revenue
Gasoline A	$\leq 30\%$	$\geq 40\%$	-	5.5
Gasoline B	$\leq 50\%$	$\geq 10\%$	-	4.5

The last column of the previous table indicates the profit (Euro/barrel). The availability of each type of oil (in barrel) and the cost (Euro/barrel) are as follows:

Oil	Availability	Cost
1	3000	3
2	2000	6
3	4000	4

Give a Linear Programming formulation for the problem of determining a mixture that maximizes the profit (difference between revenues and costs).

Solution

1.1 **Portfolio optimization**.

Parameters

- C: available capital

Decision variables

- x_1: amount of money invested in stock of type 1
- x_2: amount of money invested in stock of type 2

Model

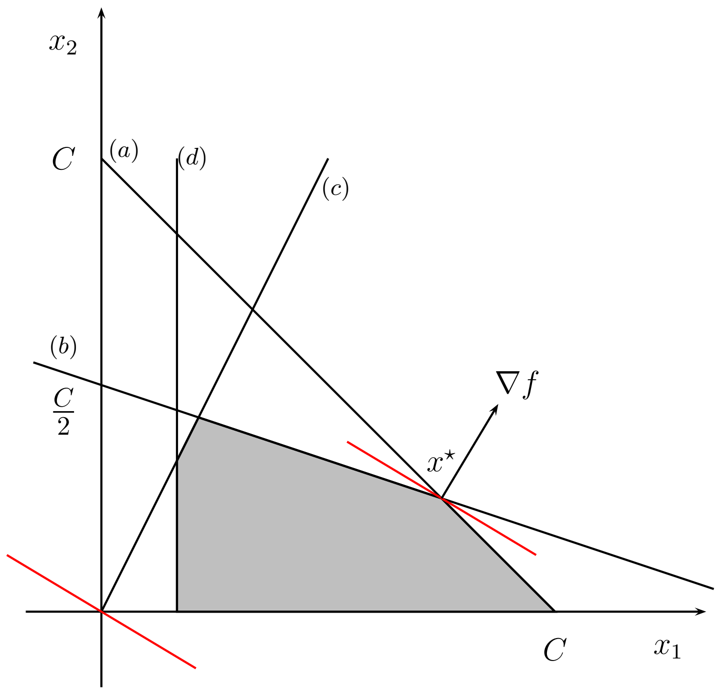
$$\begin{aligned} \max \quad & 0.15x_1 + 0.25x_2 \\ \text{s.t.} \quad & \\ & x_1 + x_2 \le C \\ & \frac{1}{3}x_1 + x_2 \le \frac{1}{2}C \\ & x_2 \le 2x_1 \\ & x_1 \ge \frac{1}{6}C \\ & x_1, x_2 \ge 0 \end{aligned}$$

To solve the problem graphically, we must identify the feasible region in $\mathbb{R}^2$ that satisfies the constraints (a) $x_2 \le -x_1 + C$, (b) $x_2 \le -\frac{1}{3}x_1 + \frac{C}{2}$, (c) $x_2 \le 2x_1$, (d) $x_1 \ge \frac{1}{6}C$, (e) $x_1 \ge 0$, and (f) $x_2 \ge 0$. For any two variables problem, any linear constraint divides $\mathbb{R}^2$ in two halfplanes.

To draw a constraint, it suffices to find any two points that satisfy it with equality (as an equation). The "border" of the constraint is then represented by the only line containing such points. For simplicity, it is convenient to choose the origin $(0, 0)$ as the first point, if feasible, and if it is not, the points $(x_1, 0)$ and $(0, x_2)$, where x_1 and x_2 are two unknowns that are to be determined w.r.t. the constraint. To identify which of the two halfplanes is the feasible one, two ways are possible. In the first one, it suffices to pick a random point and checking whether is satisfies the constraint (i.e., the inequality). If it does, the halfspace to which the point belongs is the feasible one, otherwise the other halfspace is. Alternatively, we can consider the gradient of the constraint and compare it to the direction of the inequality. For $\ge$ inequalities, the feasible halfplane is that in the direction of the gradient, while for $\le$ inequalities, it is the other one.

The feasible region is as shown in the picture. To find the feasible point where the objective function attains its maximal value, we can draw the *level curves* $f(x_1, x_2) = 0.15x_1 + 0.25x_2 = k$, where each level curve is the set of points whose objective function value is equal to k, for any constant k.

Since f is linear, the level curve $f(x_1, x_2) = k$ is a line, orthogonal to its gradient, and parametric in k. Clearly, when k is increased, we obtain parallel level lines that move towards the direction of the gradient ∇f.



Note that, by starting with $k = 0$ and by increasing it in a continuous way, the level lines of f will first intersect the feasible region at $(\frac{C}{6}, 0)$, and then, increasing k, at any other point, until the intersection is empty. The last feasible point(s) having a nonempty intersection is (are) the maximizer(s) of f over the feasible set. In this problem, since lines (a) and (b) are not parallel and the level lines are not parallel to lines (a) or (b), there is a single maximizer that corresponds to the intersection of the lines (a) and (b). The maximizer, denoted by x^*, can be found as the solution to the following (trivial) linear system

$$\begin{cases} x_2 = -x_1 + C \\ x_2 = -\frac{1}{3}x_1 + \frac{1}{2}C, \end{cases}$$

which yields $x^* = \left(\frac{3C}{4}, \frac{C}{4}\right)$, where $f^* = \frac{7C}{40}$.

1.2 Gasoline mixture

Decision variables

- x_{ij}: amount of the i-th oil used to produce the j-th gasoline, for $i \in \{1, 2, 3\}$ and $j \in \{A, B\}$
- y_j: amount of gasoline of type j-th that is produced, for $j \in \{A, B\}$

The total revenue is $5.5y_A + 4.5y_B$, the total cost, due to the amount of oil 1, 2, and 3 that is used is, respectively, $3\sum_{j\in\{A,B\}} x_{1j}$, $6\sum_{j\in\{A,B\}} x_{2j}$, and $4\sum_{j\in\{A,B\}} x_{3j}$.

Model

$$
\begin{array}{lll}
\max & 5.5y_A + 4.5y_B - 3(x_{1A} + x_{1B}) & \\
& - 6(x_{2A} + x_{2B}) - 4(x_{3A} + x_{3B}) & \text{(profit)} \\
\text{s.t.} & & \\
& x_{1A} + x_{1B} \le 3000 & \text{(availability 1)} \\
& x_{2A} + x_{2B} \le 2000 & \text{(availability 2)} \\
& x_{3A} + x_{3B} \le 4000 & \text{(availability 3)} \\
& y_A = x_{1A} + x_{2A} + x_{3A} & \text{(conservation A)} \\
& y_B = x_{1B} + x_{2B} + x_{3B} & \text{(conservation B)} \\
& x_{1A} \le 0.3y_A & \text{(min. quantity A)} \\
& x_{1B} \le 0.5y_B & \text{(min. quantity B)} \\
& x_{2A} \ge 0.4y_A & \text{(max. quantity A)} \\
& x_{2B} \ge 0.1y_B & \text{(max. quantity B)} \\
& x_{1A}, x_{2A}, x_{3A}, x_{1B}, x_{2B}, x_{3B}, y_A, y_B \ge 0 & \text{(nonnegative variables)}
\end{array}
$$

The variables y_A and y_B can be removed from the formulation by substitution: they are replaced in the objective function and in the constraints by the right-hand side terms of equations (*conservation A*) and (*conservation B*). This gives the following more compact formulation:

$$
\begin{array}{lll}
\max & 5.5(x_{1A} + x_{2A} + x_{3A}) + 4.5(x_{1B} + x_{2B} + x_{3B}) & \\
& - 3(x_{1A} + x_{1B}) - 6(x_{2A} + x_{2B}) - 4(x_{3A} + x_{3B}) & \text{(profit)} \\
\text{s.t.} & & \\
& x_{1A} + x_{1B} \le 3000 & \text{(availability 1)} \\
& x_{2A} + x_{2B} \le 2000 & \text{(availability 2)} \\
& x_{3A} + x_{3B} \le 4000 & \text{(availability 3)} \\
& x_{1A} \le 0.3(x_{1A} + x_{2A} + x_{3A}) & \text{(min. quantity A)} \\
& x_{1B} \le 0.5(x_{1B} + x_{2B} + x_{3B}) & \text{(min. quantity B)} \\
& x_{2A} \ge 0.4(x_{1A} + x_{2A} + x_{3A}) & \text{(max. quantity A)} \\
& x_{2B} \ge 0.1(x_{1B} + x_{2B} + x_{3B}) & \text{(max. quantity B)} \\
& x_{1A}, x_{2A}, x_{3A}, x_{1B}, x_{2B}, x_{3B}, y_A, y_B \ge 0 & \text{(nonnegative variables)}
\end{array}
$$

Both formulations are valid. The former one contains two additional variables and two additional constraints with respect to the latter one. In the latter formulation, the constraints are less sparse (they involve more variables due to the substitutions).

The two formulations describe a very particular instance of the gasoline mixture problem in which only two types of gasoline are produced from only three oils, for some specific values of the parameters. The models are, in a sense, not general.

When describing a problem (even though we are interested in solving a particular instance), it is advisable to give an abstract description of it, which does not depend on the data of the instance at hand.

To obtain a general formulation of the gasoline mixture problem, it suffices to define the variables and parameters of the problem as vectors and matrices, with indices belonging to well-defined sets.

Sets

- I: types of oil
- J: types of gasoline

Parameters

- c_i: cost per barrel of the i-th type of oil, for $i \in I$
- b_i: maximum availability of the i-th type of oil, for $i \in I$
- r_j: revenue per barrel of the j-th type of gasoline, for $j \in J$
- $q_{ij}^{\max}$: maximum quantity (as fraction of the unit) of the i-th type of oil in the j-th type of gasoline, for $i \in I$ and $j \in J$
- $q_{ij}^{\min}$: minimum quantity (as fraction of the unit) of the i-th type of oil in the j-th type of gasoline, for $i \in I$ and $j \in J$

Decisions variables

- x_{ij}: quantity of the i-th type of oil in the j-th type of gasoline, for $i \in I$ and $j \in J$
- y_j: quantity of the j-th type of gasoline produced, for $j \in J$

Model

$$
\begin{array}{llll}
\max & \displaystyle\sum_{j \in J} r_j y_j - \sum_{i \in I, j \in J} c_i x_{ij} & & \text{(profit)} \\
\text{s.t.} & & & \\
& \displaystyle\sum_{j \in J} x_{ij} \leq b_i & i \in I & \text{(availability)} \\
& \displaystyle y_j = \sum_{i \in I} x_{ij} & j \in J & \text{(conservation)} \\
& x_{ij} \leq q_{ij}^{\max} y_j & i \in I, j \in J & \text{(min. quantity)} \\
& x_{ij} \geq q_{ij}^{\min} y_j & i \in I, j \in J & \text{(max. quantity)} \\
& x_{ij}, y_j \geq 0 & i \in I, j \in J & \text{(non negative variables)}
\end{array}
$$

Note the differences between this formulation and the previous ones. Every "row" of this model describes a group, or family, of constraints, via the mathematical quantifiers. Observe that some constraints are defined for each combination of type of oil and type of gasoline, while, in the statement of the problem, some of those combinations where not expressed (e.g., no minimum or maximum quantities were given for oil 3). In this case, the parameters will be chosen so that these constraints are always satisfied for the specific combination of indices, e.g, by letting the minimum quantity equal to 0 and the maximum quantity equal to 1. As a simple exercise, remove each variable y_j from the formulation.