Foundations of Operations Research
Practice exercises: Linear Programming

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Exercise 1

Solve the following linear problem graphically:

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 \leq 10 \\ & x_1 \leq 4 \\ & x_2 \leq 5 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Solution

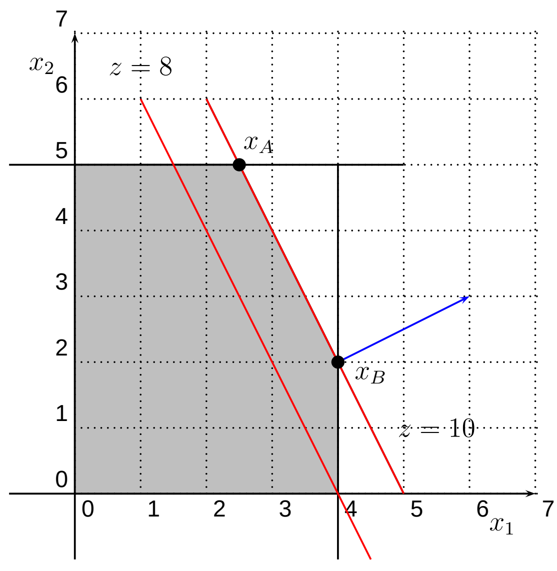

Figure 1: Polyhedron of the feasible solutions

The red lines correspond to the level curves of the objective function for $z = 8$ and $z = 10$. The blue arrow represents the gradient of the objective function. All the points laying in the segment (A, B) represent solution with value 10, and are optimal solutions. However, only $x_A = (2.5, 5)$ and $x_B = (4, 2)$ are vertices, so there are two optimal basic solutions and an infinite number of non-basic optimal solutions.

Exercise 2

Solve the following linear problem graphically:

$$\begin{aligned} \min \quad & -x_1 - x_2 \\ \text{s.t.} \quad & \\ & -3x_1 + 2x_2 \geq 6 \\ & 3x_1 + x_2 \geq 9 \\ & x_1, x_2 \geq 0. \end{aligned}$$

Solution

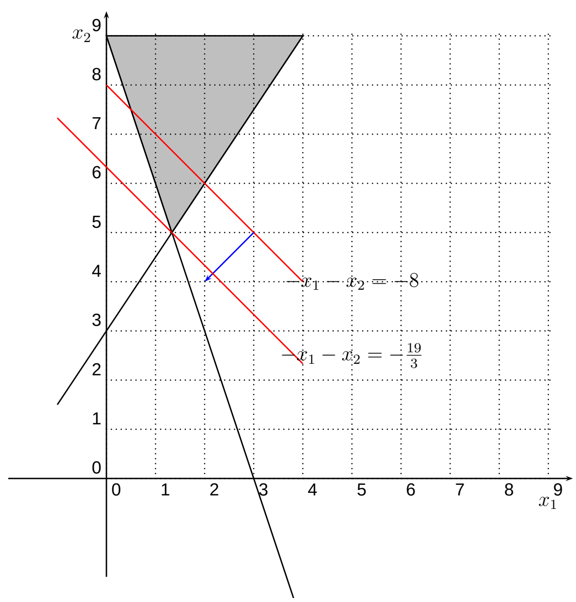

Figure 2: Polyhedron of the feasible solutions

The red lines correspond to the level curves of the objective function for $z = 8$ and $z = -\frac{19}{3}$. The blue arrow represents gradient of the objective function. The objective function can decrease indefinitely, thus the problem is unbounded.

Exercise 3

Determine using the Simplex algorithm with Bland's rule the optimal solution to the following linear programming problem:

$$\begin{aligned} \max \quad & x_1 + 3x_2 + 5x_3 + 2x_4 \\ \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 + x_4 \leq 3 \\ & 2x_1 + x_2 + x_3 + 2x_4 \leq 4 \\ & x_1, x_2, x_3, x_4 \in \mathbb{R}^+. \end{aligned}$$

Solution

The problem in standard form is:

$$\begin{aligned} \min \quad & -x_1 - 3x_2 - 5x_3 - 2x_4 \\ \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 + x_4 + s_1 = 3 \\ & 2x_1 + x_2 + x_3 + 2x_4 + s_2 = 4 \end{aligned}$$

The initial tableau is

		x_1	x_2	x_3	x_4	s_1	s_2
	0	-1	-3	-5	-2	0	0
s_1	3	1	2	3	1	1	0
s_2	4	2	1	1	2	0	1

Iteration 1: Using Bland's rule x_1 enters the basis. $\theta = \min\{\frac{3}{1}, \frac{4}{2}\} = 2$, thus s_2 exists the basis. The next tableau is

		x_1	x_2	x_3	x_4	s_1	s_2
	2	0	-2.5	-4.5	-1	0	0.5
s_1	1	0	1.5	2.5	0	1	-0.5
x_1	2	1	0.5	0.5	1	0	0.5

Iteration 2: Using Bland's rule x_2 enters the basis. $\theta = \min\{\frac{1}{1.5}, \frac{2}{0.5}\} = 0.67$, thus s_1 exists the basis. The next tableau is

		x_1	x_2	x_3	x_4	s_1	s_2
	3.67	0	0	-0.33	-1	1.67	-0.33
x_2	0.67	0	1	1.67	0	0.67	-0.33
x_1	1.67	1	0	-0.33	1	-0.33	0.67

Iteration 3: Using Bland's rule x_3 enters the basis. $\theta = \min\{\frac{0.67}{1.67}\} = 0.4$, thus x_2 exists the basis. The next tableau is

		x_1	x_2	x_3	x_4	s_1	s_2
	3.8	0	0.2	0	-1	1.8	-0.4
x_3	0.4	0	0.6	1	0	0.4	-0.2
x_1	1.8	1	0.2	0	1	-0.2	0.6

Iteration 4: Using Bland's rule x_4 enters the basis. $\theta = \min\{\frac{1.8}{1}\} = 1.8$, thus x_1 exists the basis. The next tableau is

		x_1	x_2	x_3	x_4	s_1	s_2
	5.6	1	0.4	0	0	1.6	0.2
x_3	0.4	0	0.6	1	0	0.4	-0.2
x_4	1.8	1	0.2	0	1	-0.2	0.6

All the reduced costs are nonnegative, then the optimal solution to the problem is the basis (x_1, x_4) with values:

$$x_B = \begin{bmatrix} 0.4 \\ 1.8 \end{bmatrix}$$

The value associated with the optimal solution is 5.6 (the original problem is a maximization one).

Exercise 4

Determine using the Simplex algorithm with Bland's rule the optimal solution to the following linear programming problem:

$$\begin{array}{rl} \min & -5x_1 - 2x_2 - 3x_3 - x_4 \\ \text{s.t.} & x_1 - 2x_2 + 2x_3 + 2x_4 \leq 4 \\ & -x_1 + x_2 + x_3 - x_4 \leq 6 \\ & x_i \geq 0. \end{array}$$

Solution

The problem in standard form is:

$$\begin{array}{rl} \min & -5x_1 - 2x_2 - 3x_3 - x_4 \\ \text{s.t.} & x_1 - 2x_2 + 2x_3 + 2x_4 + s_1 = 4 \\ & -x_1 + x_2 + x_3 - x_4 + s_2 = 6 \\ & x_i, s_i \geq 0. \end{array}$$

The initial tableau is

		x_1	x_2	x_3	x_4	s_1	s_2
	0	-5	-2	-3	-1	0	0
s_1	4	1	-2	2	2	1	0
s_2	6	-1	1	1	-1	0	1

Iteration 1: Using Bland's rule x_1 enters the basis. $\theta = \min\{\frac{4}{1}\} = 4$, thus s_1 exists the basis. The next tableau is

		x_1	x_2	x_3	x_4	s_1	s_2
	20	0	-12	7	9	5	0
x_1	4	1	-2	2	2	1	0
s_2	10	0	-1	3	1	1	1

Iteration 2: The only candidate variable to enter the basis is x_2. However, Since all the elements $\bar{a}[i,s]$ are negative (i.e., -2,-1), the considered problem is unbounded ($z = -\infty$).

Exercise 5

Solve the following linear programming problem using the Simplex algorithm with Bland's rule:

$$\begin{aligned} \min \quad & 3x_1 + x_2 + x_3 \\ \text{s.t.} \quad & 2x_1 + x_2 + x_3 = 6 \\ & x_1 + x_2 + 2x_3 = 2 \\ & x_1, x_2, x_3 \geq 0. \end{aligned}$$

Solution

We will execute the two-phase simplex method. In phase one we try to find a basic feasible expressed in canonical form. The auxiliary problem is:

$$\begin{aligned} \min \quad & w_1 + w_2 \\ \text{s.t.} \quad & 2x_1 + x_2 + x_3 + w_1 = 6 \\ & x_1 + x_2 + 2x_3 + w_2 = 2 \\ & x_1, x_2, x_3, w_1, w_2 \geq 0. \end{aligned}$$

We express the objective function in terms of the nonbasic variables.

$$w_1 = 6 - 2x_1 - x_2 - x_3$$

$$w_2 = 2 - x_1 - x_2 - 2x_3$$

Therefore,

$$w_1 + w_2 = 8 - 3x_1 - 2x_2 - 3x_3.$$

		x_1	x_2	x_3	w_1	w_2
	-8	-3	-2	-3	0	0
w_1	6	2	1	1	1	0
w_2	2	1	1	2	0	1

Iteration 1: Using Bland's rule x_1 enters the basis. $\theta = \min\{\frac{6}{2}, \frac{2}{1}\} = 2$, thus w_2 exists the basis. The next tableau is

		x_1	x_2	x_3	w_1	w_2
	-2	0	1	3	0	3
w_1	2	0	-1	-3	1	-2
x_1	2	1	1	2	0	1

The reduced cost are all nonnegative. The optimal solution is $[w_1, x_1] = [2, 2]$ is with an objective function value of 2. Since the objective function value of the first stage is strictly positive (and not zero) the original problem is infeasible.

Exercise 6

Consider the following linear programming problem:

$$
\begin{aligned}
\max \quad 2x_1 + x_2 \\
-2x_1 - x_2 \leq -1 \\
x_1 - x_2 \leq 3 \\
4x_1 + x_2 \leq 17 \\
x_2 \leq 5 \\
-x_1 + x_2 \leq 4
\end{aligned}
$$

where $x_1, x_2 \geq 0$.

a) Write the dual problem of the given problem.

b) Write the equations defining the complementarity slackness for the given problem (Notice that the problem and its dual are in symmetric form).

c) Exploiting the complementarity conditions say whether points (3,5) and (4,1) are optimal.

Solution

Part a)

$$
\begin{aligned}
\min \quad -y_1 + 3y_2 + 17y_3 + 5y_4 + 4y_5 \\
-2y_1 + y_2 + 4y_3 - y_5 \geq 2 \\
-y_1 - y_2 + y_3 + y_4 + y_5 \geq 1 \\
y_1, y_2, y_3, y_4, y_5 \geq 0
\end{aligned}
$$

Part b)
The equations of complementary slackness are:

$$\begin{aligned}(-2x_1 - x_2 + 1)y_1 &= 0\\(x_1 - x_2 - 3)y_2 &= 0\\(4x_1 + x_2 - 17)y_3 &= 0\\(x_2 - 5)y_4 &= 0\\(-x_1 + x_2 - 4)y_5 &= 0\\(-2y_1 + y_2 + 4y_3 - y_5 - 2)x_1 &= 0\\(-y_1 - y_2 + y_3 + y_4 + y_5 - 1)x_2 &= 0\end{aligned}$$

Part c)
Point (3,5) is feasible for the primal problem. By using the equations of complementary slackness we have that:

$$\begin{aligned}(-2x_1 - x_2 + 1)y_1 = 0 &\Rightarrow y_1 = 0\\(x_1 - x_2 - 3)y_2 = 0 &\Rightarrow y_2 = 0\\(-x_1 + x_2 - 4)y_5 = 0 &\Rightarrow y_5 = 0\\x_1 > 0 &\Rightarrow 4y_3 = 2\\x_2 > 0 &\Rightarrow y_3 + y_4 = 1\end{aligned}$$

We get that $y_3 = \dfrac{1}{2}, y_4 = \dfrac{1}{2}$, which is a feasible solution to the dual problem and thus (3,5) is optimal.

Point (4,1) is feasible for the primal problem. By using the equations of complementary slackness we have that:

$$\begin{aligned}(-2x_1 - x_2 + 1)y_1 = 0 &\Rightarrow y_1 = 0\\(x_2 - 5)y_4 = 0 &\Rightarrow y_4 = 0\\(-x_1 + x_2 - 4)y_5 = 0 &\Rightarrow y_5 = 0\\x_1 > 0 &\Rightarrow y_2 + 4y_3 = 2\\x_2 > 0 &\Rightarrow -y_2 + y_3 = 1\end{aligned}$$

We get that $y_2 = -\frac{2}{5}, y_3 = \frac{3}{5}$, which is not a feasible solution to the dual problem and thus (4,1) is not optimal.

Exercise 7

Consider the following problem:

$$\max z = 9x_1 + 8x_2$$

$$\begin{aligned} x_1 - 2x_2 &\leq -1 \\ 4x_1 + 3x_2 &\leq 4 \\ -x_1 + 2x_2 &\leq 3 \\ 2x_1 - x_2 &\leq -4 \end{aligned}$$

Verify if solution $x_1 = -3$, $x_2 = -1$ is optimal. Verify if solution $x_1 = -\frac{5}{3}$, $x_2 = \frac{2}{3}$ is optimal.

Solution

The dual problem is:

$$\min z_D = -y_1 + 4y_2 + 3y_3 - 4y_4$$

$$\begin{aligned} y_1 + 4y_2 - y_3 + 2y_4 &= 9 \\ -2y_1 + 3y_2 + 2y_3 - y_4 &= 8 \\ y_i &\geq 0 \ \forall i \end{aligned}$$

The equations of complementary slackness are:

$$\begin{aligned} (x_1 - 2x_2 + 1)y_1 &= 0 \\ (4x_1 + 3x_2 - 4)y_2 &= 0 \\ (-x_1 + 2x_2 - 3)y_3 &= 0 \\ (2x_1 - x_2 + 4)y_4 &= 0 \\ (y_1 + 4y_2 - y_3 + 2y_4 - 9)x_1 &= 0 \\ (-2y_1 + 3y_2 + 2y_3 - y_4 - 8)x_2 &= 0 \end{aligned}$$

Point (-3,-1) is feasible for the primal problem ($z = -35$). By using the equations of complementary slackness we infer that the dual variable y_2 , y_3 and y_4 are equal to 0. Therefore,

$$\begin{aligned} y_1 &= 9 \\ -2y_1 &= 8 \end{aligned}$$

This system has no feasible solution. Therefore the solution $x_1 = -3, x_2 = -1$ is not optimal.

Point $(-\frac{5}{3}, \frac{2}{3})$ is feasible for the primal problem ($z = -\frac{29}{3}$). By using the equations of complementary slackness we infer that the dual variable y_1 and y_2 are equal to 0.
Considering values $x_1 = -\frac{5}{3}$, $x_2 = \frac{2}{3}$ ($z = -\frac{29}{3}$), constraints (3) and (4) hold with equality. Dual variable y_1 and y_2 are equal to 0 for the condition of complementary slackness. Therefore,

$$\begin{aligned} -y_3 + 2y_4 &= 9 \\ +2y_3 - y_4 &= 8 \end{aligned}$$

We get that $y_3 = \dfrac{25}{3}, y_4 = \dfrac{26}{3}$, which is a feasible solution to the dual problem and thus $(-\frac{5}{3}, \frac{2}{3})$ is optimal.

Exercise 8

Consider the following Linear Programming problem:

$$\begin{aligned} \min -x_1 + 2x_2 & & (1) \\ -x_1 + x_2 & \leq 2 & (2) \\ x_1 + x_3 & = 3 & (3) \\ 2x_1 + x_2 & \geq 1 & (4) \\ 2x_1 - 6x_2 & \leq 15 & (5) \\ x_1, x_3 \geq 0, & \quad x_2 \text{ free} & (6) \end{aligned}$$

Without applying any problem transformation, write the dual problem and the complementary slackness conditions (for both problems).

Consider two solutions $x^1 = \left[3, -\frac{3}{2}, 0\right]$ and $x^2 = \left[\frac{3}{2}, -2, \frac{3}{2}\right]$. Determine the dual complementary solutions and discuss the optimality of both primal solutions.

Solution

$$\begin{aligned}
\max \quad & 2y_1 + 3y_2 + y_3 + 15y_4 & \\
& -y_1 + y_2 + 2y_3 + 2y_4 & \leq -1 \\
& y_1 + y_3 - 6y_4 & = 2 \\
& y_2 & \leq 0 \\
& y_1, y_4 & \leq 0 \\
& y_2 & free \\
& y_3 & \geq 0
\end{aligned}$$

Note that $y_2 \leq 0$ dominates the constraint that y_2 is free. Therefore, the constraint that y_2 is free is not necessary.
The complementary slackness conditions are the following:

$$\begin{aligned}
(-x_1 + x_2 - 2)y_1 &= 0 \\
(x_1 + x_3 - 3)y_2 &= 0 \\
(2x_1 + x_2 - 1)y_3 &= 0 \\
(2x_1 - 6x_2 - 15)y_4 &= 0 \\
(-y_1 + y_2 + 2y_3 + 2y_4 + 1)x_1 &= 0 \\
(y_1 + y_3 - 6y_4 - 2)x_2 &= 0 \\
y_2 x_3 &= 0
\end{aligned}$$

Point $x^1 = \left[3, -\frac{3}{2}, 0\right]$ is feasible for the primal problem ($z = -6$). By using the equations of complementary slackness we infer that:

- Constraint 1: $-(3) + (-\frac{3}{2}) - 2 = -\frac{13}{2} \Longrightarrow y_1 = 0$
- Constraint 2: $(3 - 3) = 0$
- Constraint 3: $(2 \cdot 3 - \frac{3}{2} - 1) = \frac{7}{2} \Longrightarrow y_3 = 0$
- Constraint 4: $(2 \cdot 3 + 6 \cdot \frac{3}{2} - 15) = 0.$

Dual variable y_1 and y_3 are equal to 0 due to the condition of complementary slackness. Therefore,

$$\begin{aligned}
y_2 + 2y_4 &= -1 \\
-6y_4 &= 2
\end{aligned}$$

We get $y_2 = -\frac{1}{3}, y_4 = -\frac{1}{3}$, which is a feasible solution to the dual problem and thus $x^1 = \left[3, -\frac{3}{2}, 0\right]$ is optimal.

Point $x^2 = \left[\frac{3}{2}, -2, \frac{3}{2}\right]$ is feasible for the primal problem ($z = -\frac{11}{2}$). By using the equations of complementary slackness we infer that:

- Constraint 1: $-(\frac{3}{2}) + (-2) - 2 = -\frac{11}{2} \Longrightarrow y_1 = 0$
- Constraint 2: $(\frac{3}{2} + \frac{3}{2} - 3) = 0$
- Constraint 3: $(2 \cdot \frac{3}{2} - 2 - 1) = 0$
- Constraint 4: $(2 \cdot \frac{3}{2} + 6 \cdot 2 - 15) = 0$.

$x_3 > 0$ and $y_2 x_3 = 0$, therefore $y_2 = 0$. The system of dual constraints can be rewritten as:

$$\begin{aligned} 2y_3 + 2y_4 &= -1 \\ y_3 - 6y_4 &= 2 \end{aligned}$$

The solution to the above system yields $y_4 = -\frac{5}{14}$ $y_3 = -\frac{1}{7}$, which is not feasible for the dual problem ($y_3 \geq 0$), therefore the considered solution is not optimal.

Exercise 9

Consider the following Linear Programming problem:

$$\begin{aligned} \min -2x_1 + 2x_2 - 2x_3 & \\ 2x_1 - 2x_2 - x_3 &\leq 2 \\ -3x_1 + 3x_2 + 2x_3 &\leq 3 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

- Write the dual problem.
- Solve the primal problem using the Simplex method with Bland's rule.

Solution

The dual problem is:

$$\begin{aligned} \max \quad 2y_1 + 3y_2 & \\ 2y_1 - 3y_2 &\leq -2 \\ -2y_1 + 3y_2 &\leq 2 \\ -y_1 + 2y_2 &\leq -2 \\ y_1 &\leq 0 \\ y_2 &\leq 0 \end{aligned}$$

The problem in standard form is:

$$\begin{aligned} \min -2x_1 + 2x_2 - 2x_3 & \\ 2x_1 - 2x_2 - x_3 + s_1 &= 2 \\ -3x_1 + 3x_2 + 2x_3 + s_2 &= 3 \\ x_1, x_2, x_3, s_1, s_2 &\geq 0 \end{aligned}$$

The initial tableau is

		x_1	x_2	x_3	s_1	2_2
	0	-2	2	-2	0	0
s_1	2	2	-2	-1	1	0
s_2	3	-3	3	2	0	1

Iteration 1: Using Bland's rule x_1 enters the basis. $\theta = \min\{\frac{2}{2}\} = 1$, thus s_1 exists the basis. The next tableau is

		x_1	x_2	x_3	s_1	2_2
	2	0	0	-3	1	0
x_1	1	1	-1	-0.5	0.5	0
s_2	6	0	0	0.5	1.5	1

Iteration 2: x_3 enters the basis. $\theta = \min\{\frac{6}{0.5}\} = 12$, thus s_2 exists the basis. The next tableau is

		x_1	x_2	x_3	s_1	2_2
	38	0	0	0	10	6
x_1	7	1	-1	0	2	1
x_3	12	0	0	1	3	2

All the reduced costs are nonnegative, then the optimal solution to the problem is the basis (x_1, x_3) with values:

$$x_B = \begin{bmatrix} 7 \\ 12 \end{bmatrix}$$

The value associated with the optimal solution is -38.