Example 1:

Find an optimal solution of the ILP problem

$$
\begin{aligned}
z_{ILP} = \max \quad & 4x_1 - x_2 \\
\text{s.t.} \quad & 4x_1 + 2x_2 \leq 19 \\
& 10x_1 - 4x_2 \leq 25 \\
& x_2 \leq \tfrac{9}{2} \\
& x_1, x_2 \in \mathbb{Z}^+
\end{aligned}
$$

by Branch and Bound, solving the linear relaxations of the subproblems graphically.

The feasible region of the linear relaxation is reported in Figure 1.

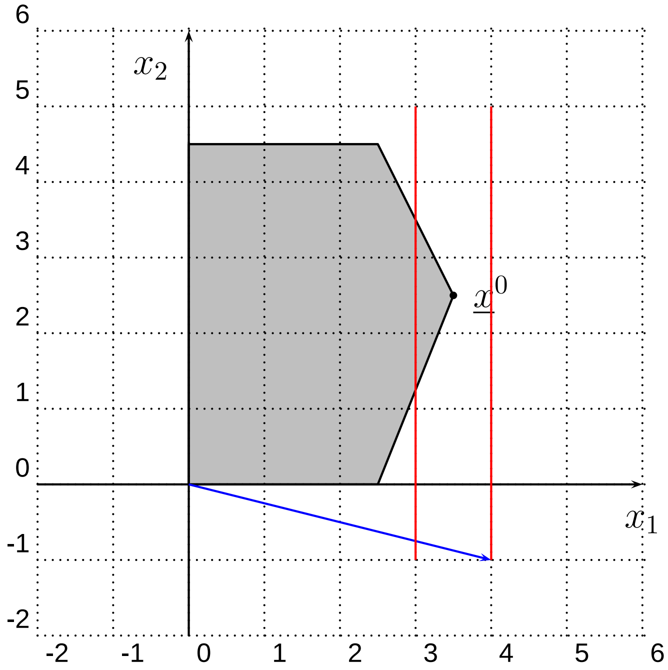


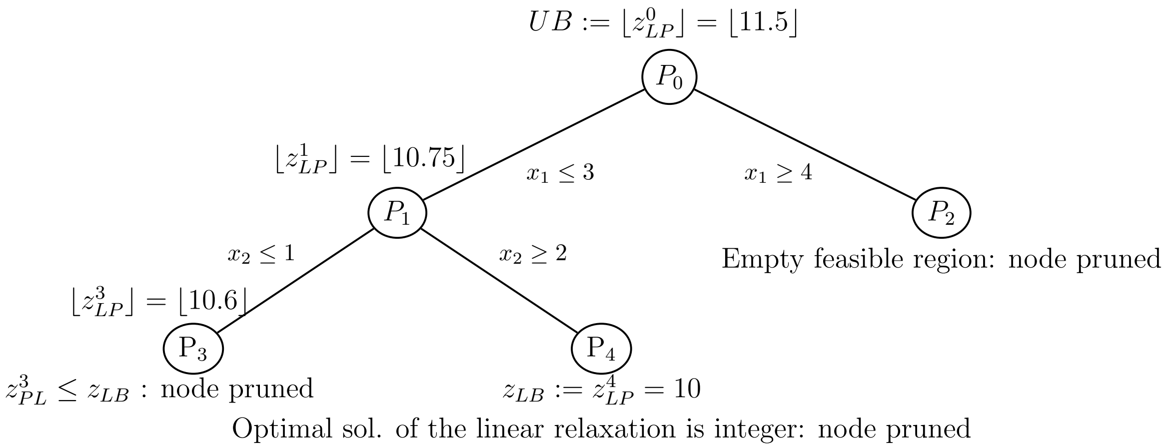
Figura 1: Feasible region of the linear relaxation of the ILP

Optimal solution of the linear relaxation: $\underline{x}^0 = \left(\frac{7}{2}, \frac{5}{2}\right)$

Its value $z^0_{LP} = \frac{23}{2} = 11.5$ provides and upper bound $\lfloor z^0_{LP} \rfloor = \lfloor 11.5 \rfloor$ on z_{ILP}.

First branching step w.r.t. x_1. The branching constraints $x_1 \leq 3$ and $x_1 \geq 4$ are represented in red in Figure 1.

The Branch and Bound enumeration tree is as follows:



The details concerning all subproblems are reported in Figure 2, 3 e 4.

The optimal solution $\underline{x}^* = (3, 2)$ with $z^* = 10$ is found in subproblem (node) P_4.

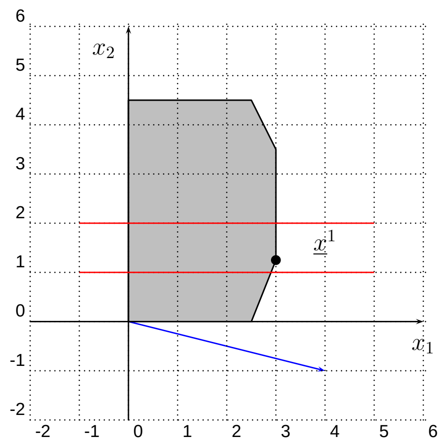


Figura 2: Subproblem P_1: optimal solution of the linear relaxation $\underline{x}^1 = \left(3, \frac{5}{4}\right)$

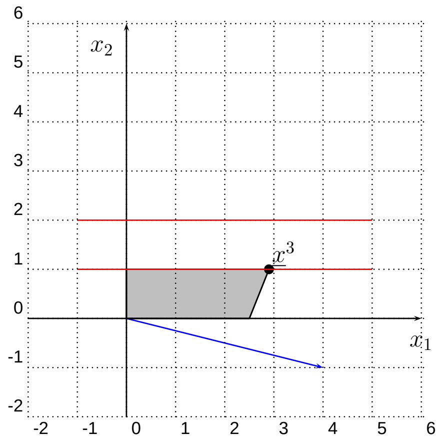


Figura 3: Subproblem P_3: optimal solution of the linear relaxation $\underline{x}^3 = \left(\frac{29}{10}, 1\right)$

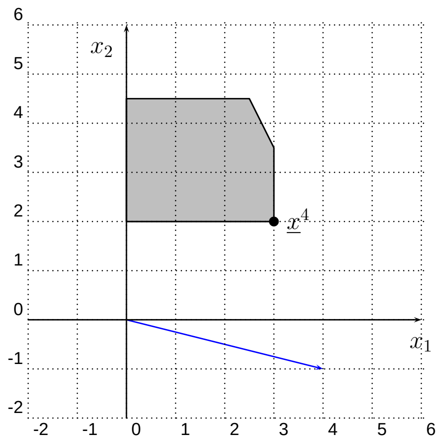


Figura 4: Subproblem P_4: optimal solution of the linear relaxation $\underline{x}^4 = (3, 2)$

Example 2:

Consider the binary knapsack problem

$$\begin{aligned} z_{PLI} = \max \quad & 10x_1 + 12x_2 + 5x_3 + 7x_4 + 9x_5 \\ \text{s.t.} \quad & 5x_1 + 8x_2 + 6x_3 + 2x_4 + 7x_5 \leq 14 \\ & x_1, \ldots, x_5 \in \{0, 1\}. \end{aligned}$$

Its linear relaxation is obtained by just substituting the integrality constraints $x_i \in \{0, 1\}$, $1 \leq i \leq 5$, with $0 \leq x_i \leq 1$, for $1 \leq i \leq 5$. To determine an optimal solution of the linear relaxation it is not necessary to apply a linear programming algorithm (e.g., simplex or interior point methods). A simple greedy procedure suffices.

Greedy procedure for fractional knapsack:

Order the variables (items) according to nonincreasing profit-to-weight ratios p_i/w_i:

var.	x_4	x_1	x_2	x_5	x_3
p_i	7	10	12	9	5
w_i	2	5	8	7	6
p_i/w_i	3.5	2.0	1.5	1.2	0.8

Consider the items in that order, and for each item try to include in the knapsack the largest possible fraction of it, based on the residual knapsack capacity. Stop when the knapsack is full.

The branching tree is reported in Figure 5. The details of all subproblems (nodes) are as follows.

Node 0: The optimal solution of the linear relaxation is $x_4 = 1$, $x_1 = 1$, $x_2 = \frac{7}{8}$, $x_5 = x_3 = 0$. Thus

$$UB := u^0 = 7 + 10 + \left\lfloor 12 \cdot \frac{7}{8} \right\rfloor = 27,$$

the feasible integer solution $x_4 = 1$, $x_1 = 1$, $x_2 = 0$, $x_5 = x_3 = 0$ provides a lower bound $l^0 = 17$ on z_{PLI} and we set $LB := 17$.

Node 1: At node 1 we set $x_2 = 1$. An optimal solution of the linear relaxation is $x_2 = 1$, $x_4 = 1$, $x_1 = \frac{4}{5}$, $x_5 = x_3 = 0$, hence

$$u^1 = 12 + 7 + \left\lfloor 10 \cdot \frac{4}{5} \right\rfloor = 27, \quad l^1 = 19 \text{ and we update } LB := 19.$$

Node 2: At node 2 we set $x_2 = 0$. An optimal solution of the linear relaxation is $x_4 = x_1 = x_5 = 1$, $x_2 = x_3 = 0$, which is integer. Thus

$$u^2 = l^2 = 7 + 10 + 9 = 26 \text{ and we update } LB := 26.$$

Node 3: At node 3 we set $x_2 = x_1 = 1$. Together both items use $8 + 5 = 13$ units of space. None of the other items can be fully inserted in the residual space of $14 - 13 = 2$ units. Since the subregion associated with this node does not contain other feasible solutions, the node can be fathomed (closed). The only feasible solution $x_2 = x_1 = 1$ $x_3 = x_4 = x_5 = 0$ has a value of 22. Since $u^3 = l^3 = 22 < LB = 26$, we do not update LB.

Node 4: At node 4 we set $x_2 = 1$, $x_1 = 0$. The optimal solution of the linear relaxation $x_2 = x_4 = 1$, $x_5 = \frac{4}{7}$, $x_1 = x_3 = 0$. Since

$$u^4 = 12 + 7 + \left\lfloor 9 \cdot \frac{4}{7} \right\rfloor = 24 < LB = 26,$$

the node can be fathomed (closed).

The procedure (search) ends when all nodes are fathomed (closed). $x_1 = x_4 = x_5 = 1$,$x_2 = x_3 = 0$ turns out to be an optimal solution with $z_{PLI} = 26$.

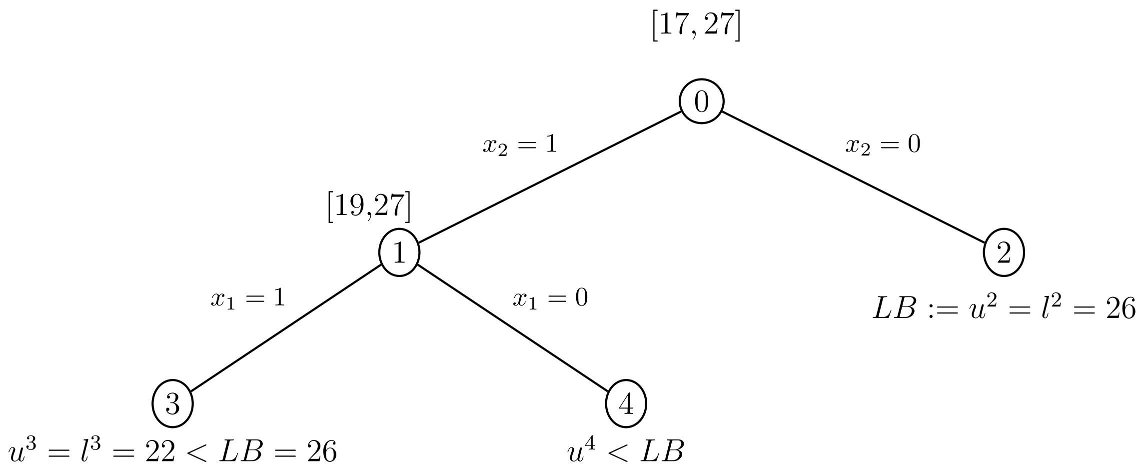


Figura 5: Branch and bound tree for the knapsack problem